Probability, homework 11, due November 22.

Exercise 1. For fixed $p, q \in [0,1]$, consider a Markov chain X with two states $\{1,2\}$, with transition matrix

$$\pi = (\pi(i,j))_{1\le i,j\le 2} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

(i) For which p, q is the chain irreducible? Aperiodic?
(ii) What are the invariant probability measures of X?
(iii) Compute $\pi^{(n)}$, $n \ge 1$.
(iv) When X is irreducible, for this invariant probability measure μ, calculate

$$d_1(n) := \frac{1}{2}\left(|\mathbb{P}_1(X_n = 1) - \mu(1)| + |\mathbb{P}_1(X_n = 2) - \mu(2)|\right)$$
$$d_2(n) := \frac{1}{2}\left(|\mathbb{P}_2(X_n = 1) - \mu(1)| + |\mathbb{P}_2(X_n = 2) - \mu(2)|\right)$$

where $\mathbb{P}_x$ means the chain starts at x.

Exercise 2. Consider a Markov chain X with state space $\mathbb{N}$ and transition matrix

$$\pi(0,0) = r_0,\ \pi(0,1) = p_0, \text{and } \forall i \ge 1,\ \pi(i,i-1) = q_i,\ \pi(i,i) = r_i,\ \pi(i,i+1) = p_i,$$

with $p_0, r_0 > 0$, $p_0 + r_0 = 1$ and for all $i \ge 1$, $p_i, q_i > 0$, $p_i + q_i + r_i = 1$. Prove that the chain is irreducible, aperiodic. Give a necessary and sufficient condition for the chain to have an invariant probability measure.

Exercise 3. Consider a Markov chain X with state space $\{0, 1, \dots, n\}$ and transition matrix

$$\pi(0,k) = \frac{1}{2^{k+1}},\ 0 \le k \le n-1,\ \pi(0,n) = \frac{1}{2^n}$$
$$\pi(k,k-1) = 1,\ 1 \le k \le n-1,\ \pi(n,n) = \pi(n,n-1) = \frac{1}{2}.$$

(i) Prove that the chain has a unique invariant probability measure μ and calculate it.
(ii) Prove that for any $0 \le x_0 \le n-1$, $\pi^{(x_0+1)}(x_0, \cdot) = \mu$.
(iii) Prove that for any $0 \le x_0 \le n$, $\pi^{(n)}(x_0, \cdot) = \mu$.
(iv) For any $t \ge 1$, calculate

$$d(t) := \frac{1}{2}\sum_{x=0}^{n} \left|\pi^{(t)}(n,x) - \mu(x)\right|,$$

and plot $t \mapsto d(t)$.

Exercise 4. Let $(G, \cdot)$ be a group, μ a probability measure on G and X the Markov chain such that $\pi(g, h\cdot g) = \mu(h)$. We call such a process X a random walk on G with jump kernel μ.

(i) Explain why the usual random walk on $\mathbb{Z}^d$ is such process. Same question for the usual random walk on $(\mathbb{Z}/n\mathbb{Z})^d$, $n \ge 1$.
(ii) Consider the following shuffling of a deck of $n \ge 2$ cards: pick two such distinct cards uniformly at random and exhange their positions in the deck. Show that this is also an example of a random walk on a group.
(iii) Let $\mathcal{H} = \{h_1 \cdot h_2 \cdot \dots \cdot h_n, \mu(h_i) > 0, 1 \le i \le n, n \in \mathbb{N}\}$. Discuss irreductibility of X depending on $\mathcal{H}$.

(iv) If X is irreducible on finite G, what are the invariant probability measures? What if G is not finite?

(v) Make some search to define a reversible Markov chain. In the context of this exercise, show that X is reversible if and only if $\mu(h) = \mu(h^{-1})$ for any $h \in G$.

(vi) Give an example of an irreducible random walk on a group which is not reversible.

Exercise 5. An ant walks on a round clock, starting at 0, up to the moment it has visited all numbers. At each second, it walks either clockwise or counterclockwise, with probability 1/2 to a neighbouring number, and through independent steps. Let X be the final position of the ant. Prove it is equidistributed on $\{1, 2, \ldots, 11\}$.