Probability, homework 10, due November 15.

This week's homework is short, so that you have time to read again carefully what you learned on conditionning, i.e. Sections 4.1, 4.2, 4.3 in Varadhan's book.

Exercise 1. Give an example of random variables X, Y, Z such that (X, Y), (X, Z), and (Y, Z) are Gaussian vectors, but not (X, Y, Z).

Exercise 2. Let $(p_n)_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$ converging to $p \in (0, 1)$. Let Y_n be a random variable, binomial with parameters n and p_n: Y_n is equal in distribution to the sum of n independent Bernoulli random variables with parameter p_n. State and prove a central limit theorem for Y_n.

Exercise 3. On the same probability space, let X and Y be two bounded random variables , i.e. there exists $C > 0$ such that $|X(\omega)| + |Y(\omega)| < C$ for any $\omega \in \Omega$. Prove that X and Y are independent if and only if for any $k, \ell \in \mathbb{N}$, we have

$$\mathbb{E}[X^k Y^\ell] = \mathbb{E}[X^k] \cdot \mathbb{E}[Y^\ell].$$

Exercise 4. *The number of buses stopping till time t.* Let $(X_n)_{n\geq 1}$ be i.i.d, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, X_1 being an exponential random variable with parameter 1. Define $T_0 = 0$, $T_n = X_1 + \cdots + X_n$, and for any $t > 0$,

$$N_t = \max\{n \geq 0 \mid T_n \leq t\}$$

a) For any $n \geq 1$, calculate the joint distribution of $(T_1, \ldots, T_n)$.
b) Deduce the distribution of N_t, for arbitrary t.