Probability, homework 8, due November 22.

Exercise 1. For fixed $p, q \in [0,1]$, consider a Markov chain X with two states $\{1,2\}$, with transition matrix

$$\pi = (\pi(i,j))_{1\leq i,j\leq 2} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

(i) For which p, q is the chain irreducible? Aperiodic (see Lemma 4.14 and Remark 4.10 in Varadhan)?
(ii) What are the invariant probability measures of X?
(iii) Compute $\pi^{(n)}$, $n \geq 1$.
(iv) When X is irreducible, for this invariant probability measure μ, calculate

$$d_1(n) := \frac{1}{2}\left(|\mathbb{P}_1(X_n = 1) - \mu(1)| + |\mathbb{P}_1(X_n = 2) - \mu(2)|\right)$$
$$d_2(n) := \frac{1}{2}\left(|\mathbb{P}_2(X_n = 1) - \mu(1)| + |\mathbb{P}_2(X_n = 2) - \mu(2)|\right)$$

where $\mathbb{P}_x$ means the chain starts at x.

Exercise 2. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\mathcal{F}_T$ is a σ-field.

Exercise 3. Let S and T be stopping times for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\max(S,T)$ and $\min(S,T)$ are stopping times.

Exercise 4. Let $S \leq T$ be two stopping times and $A \in \mathscr{F}_S$. Define $U(\omega) = S(\omega)$ if $\omega \in A$, $U(\omega) = T(\omega)$ if $\omega \notin A$. Prove that U is a stopping time.

Exercise 5. An ant walks on a round clock, starting at 0, up to the moment it has visited all numbers. At each second, it walks either clockwise or counterclockwise, with probability 1/2 to a neighbouring number, and through independent steps. Let X be the final position of the ant. Prove it is equidistributed on $\{1, 2, \dots, 11\}$.

Exercise 6. Let $X_1, X_2, \dots$ be i.i.d., $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, and $S_n = X_1 + \cdots + X_n$. Prove that the following random variable converges in distribution as $n \to \infty$, and identify the limit:

$$\left(\sum_{k=1}^{n} e^{S_k}\right)^{\frac{1}{\sqrt{n}}}.$$