Probability, homework 7, due November 8.

Exercise 1. Let X and Y be independent Gaussian random variables with null expectation and variance 1. Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are also independent $\mathcal{N}(0,1)$.

Exercise 2. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d random variables, with uniform distribution on $[0,1]$. Let $Y_n = (X_n)^n$.

(1) Calculate the distribution of Y_n.
(2) Show that $(Y_n)_{n\geq 0}$ converges to 0 in probability.
(3) Show that $(Y_n)_{n\geq 0}$ converges in L^1.
(4) Show that almost surely $(Y_n)_{n\geq 0}$ does not converge.

Exercise 3. Let $(X_n)_{n\geq 1}$ be i.i.d. Bernoulli random variables with parameter $p \in (0,1)$, i.e. $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. Let N be a Poisson random variable with parameter $\lambda > 0$, i.e. for any $k \geq 0$ we have $\mathbb{P}(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$. Assume N is independent from $(X_n)_{n\geq 1}$.

Let $P = \sum_{i=1}^{N} X_i$, $F = N - P$.

a) What is the joint distribution of (P,N)?

b) Prove that P and F are independent.

Exercise 4. *The number of buses stopping till time t.* Let $(X_n)_{n\geq 1}$ be i.i.d, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, X_1 being an exponential random variable with parameter 1. Define $T_0 = 0$, $T_n = X_1 + \cdots + X_n$, and for any $t > 0$,

$$N_t = \max\{n \geq 0 \mid T_n \leq t\}$$

a) For any $n \geq 1$, calculate the joint distribution of $(T_1, \ldots, T_n)$.

b) Deduce the distribution of N_t, for arbitrary t.

Exercise 5. Let $(X_n)_{n\geq 0}$ be real, independent, random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that the radius of convergence R of the random series $\sum_{n\geq 0} X_n z^n$ is almost surely constant.

b) Assume also that the X_n's have the same distribution. Prove that $R = 0$ a.s. if $\mathbb{E}[\log(|X_0|)_+] = \infty$, and $R \geq 1$ a.s. if $\mathbb{E}[\log(|X_0|)_+] < \infty$.

Exercise 6. Prove that there is no probability measure on $\mathbb{N}$ such that for any $n \geq 1$, the probability of the set of multiples of n is $1/n$.

Exercise 7. Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}, \mathcal{H}$ sub σ-fields of $\mathcal{F}$ such that $\sigma(\mathcal{G}, \mathcal{H}) = \mathcal{F}$. Find counterexamples to the following assertions:

(i) If $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ then X and Y are independent.
(ii) If $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}] = 0$ then $X = 0$.
(iii) If X and Y are independent then so are $\mathbb{E}[X \mid \mathcal{G}]$ and $\mathbb{E}[Y \mid \mathcal{G}]$.

Exercise 8. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Show that $|\mathbb{E}(Y \mid \mathcal{G})| \leq \mathbb{E}(|Y| \mid \mathcal{G})$ (almost surely).

Exercise 9. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-field of $\mathcal{G}$. Show that $\mathbb{E}(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(Y \mid \mathcal{H})$ (almost surely).

Exercise 10. Let $(X_n)_{n\geq 0}$ be defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Assume this sequence converges in probability (under $\mathbb{P}$) to X. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{A})$ assumed to be absolutely continuous w.r.t. $\mathbb{P}$. Prove that $X_n \to X$ in probability under $\mathbb{Q}$.