Probability, homework 6, due October 25.

Exercise 1. Assume $(\Omega, \mathcal{A}, \mathbb{P})$ is such that Ω is countable and $\mathcal{A} = 2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 2. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 3. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Exercise 4. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^n X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Exercise 5. Calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Exercise 6. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 7. A sequence of random variables $(X_i)_{i\geq 1}$ is said to be completely convergent to X if for any $\varepsilon > 0$, we have $\sum_{i\geq 1} \mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that complete convergence implies almost sure convergence.

Exercise 8. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_\ell) = \mu$ for any ℓ, and a weak correlation in the following sense: $\mathrm{Cov}(X_k, X_\ell) \leq f(|k - \ell|)$ for all indexes k, ℓ, where the sequence $(f(m))_{m\geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1} \sum_{k=1}^n X_k)_{n\geq 1}$ converges to μ in L^2.

Long problem. The goal is to prove the Erdős-Kac theorem: if $w(m)$ denotes the number of distinct prime factors of m and k is a random variable uniformly distributed on $[\![1, n]\!]$, then the following convergence in distribution holds:

$$\frac{w(k) - \log\log n}{\sqrt{\log\log n}} \underset{n\to\infty}{\longrightarrow} \mathscr{N}(0,1).$$

(i) Prove that if $(X_n)_{n\geq 1}$ converges in distribution to $\mathscr{N}(0,1)$ and $\sup_{n\geq 1} \mathbb{E}[X_n^{2k}] < \infty$ for any $k \in \mathbb{N}$, then $\lim_{n\to\infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathscr{N}(0,1)^k]$ for any $k \in \mathbb{N}$.

(ii) Prove that for any $x \in \mathbb{R}$ and $d \geq 1$ we have
$$\left| e^{\mathrm{i}x} - \sum_{\ell=0}^{d} \frac{(\mathrm{i}x)^\ell}{\ell!} \right| \leq \frac{|x|^{d+1}}{(d+1)!}.$$

(iii) Assume that $\lim_{n\to\infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathscr{N}(0,1)^k]$ for any $k \in \mathbb{N}$. Prove that X_n converges in distribution to X.

(iv) Let $w_y(m)$ be the number of prime factors of m which are smaller than y. Let $(B_p)_{p \text{ prime}}$ be independent random variables such that $\mathbb{P}(B_p = 1) = 1 - \mathbb{P}(B_p = 0) = \frac{1}{p}$, $W_y = \sum_{p \leq y} B_p$, $\mu_y = \sum_{p\leq y} \frac{1}{p}$, $\sigma_y^2 = \sum_{p \leq y}(\frac{1}{p} - \frac{1}{p^2})$. Prove that if $y = n^{\mathrm{o}(1)}$, then for any $d \in \mathbb{N}$ we have
$$\mathbb{E}\left[\left(\frac{w_y(k) - \mu_y}{\sigma_y}\right)^d\right] - \mathbb{E}\left[\left(\frac{W_y - \mu_y}{\sigma_y}\right)^d\right] \to 0$$
as $n \to \infty$.

(v) Conclude.