Probability, homework 5, due October 18.

Exercise 1. Prove that if a sequence of real random variables (X_n) converge in distribution to X, and (Y_n) converges in distribution to a constant c, then X_n+Y_n converges in distribution to $X+c$.

Exercise 2. Assume that (X,Y) has joint density

$$ce^{-(1+x^2)(1+y^2)},$$

where c is properly chosen. Are X and Y Gaussian random variables? Is (X,Y) a Gaussian vector?

Exercise 3. Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables, with X_i uniform on $[-i,i]$. Let $S_n = X_1+\dots+X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

Exercise 4. Find a probability distribution μ of a $\mathbb{Z}$-valued random variable which is symmetric ($\mu(\{i\})=\mu(\{-i\})$ for any $i\in\mathbb{Z}$), not integrable, but such that its characteristic function is differentiable at 0.

Exercise 5. For any probability measure μ supported on $\mathbb{R}_+$, one defines the Laplace transform as

$$\mathscr{L}_\mu(\lambda)=\int_0^\infty e^{-\lambda x}\mathrm{d}\mu(x),\ \lambda\geq 0.$$

(1) Prove that $\mathscr{L}_\mu$ is well-defined, continuous on $\mathbb{R}_+$ and $\mathscr{C}^\infty$ on $\mathbb{R}_+^*$.
(2) Prove that $\mathscr{L}_\mu$ characterizes the probability measure μ supported on $\mathbb{R}_+$.
(3) Assume that for a sequence $(\mu_n)_{n\geq 1}$ of probability measure supported on $\mathbb{R}_+$, one has $\mathscr{L}_{\mu_n}(\lambda)\to\ell(\lambda)$ for any $\lambda\geq 0$, and ℓ is right-continuous at 0. Prove that $(\mu_n)_{n\geq 1}$ is tight, and that it converges weakly to a measure μ such that $\ell=\mathscr{L}_\mu$.

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for $X_1,X_2\dots$ i.i.d. standard Gaussian random variables, denoting $S_n=X_1+\dots+X_n$, we have

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=1\right)=1 \tag{0.1}$$

(1) Prove that

$$\mathbb{P}(X_1>\lambda)\underset{\lambda\to\infty}{\sim}\frac{1}{\lambda\sqrt{2\pi}}e^{-\frac{\lambda^2}{2}}.$$

In the following questions we denote $f(n)=\sqrt{2n\log\log n}$, $\lambda>1$, $c,\alpha>0$, $A_k=\{S_{\lfloor\lambda^k\rfloor}\geq cf(\lambda^k)\}$, $C_k=\{S_{\lfloor\lambda^{k+1}\rfloor}-S_{\lfloor\lambda^k\rfloor}\geq cf(\lambda^{k+1}-\lambda^k)\}$ and $D_k=\{\sup_{n\in[\![\lambda^k,\lambda^{k+1}]\!]}\frac{S_n-S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)}\geq\alpha\}$.

(2) Prove that for any $c>1$ we have $\sum_{k\geq 1}\mathbb{P}(A_k)<\infty$ and

$$\limsup_{k\to\infty}\frac{S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)}\leq 1 \text{ a.s.}$$

(3) Prove that for any $c < 1$ we have $\sum_{k \geq 1} \mathbb{P}(C_k) = \infty$ and

$$\mathbb{P}(C_k \text{ i.o.}) = 1.$$

(4) Let $\varepsilon > 0$ and choose $c = 1 - \varepsilon/10$. Prove that almost surely the following inequality holds for infinitely many k:

$$\frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq c \frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} - (1+\varepsilon) \frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) By choosing a large enough λ in the previous inequality, prove that almost surely

$$\limsup_{n \to \infty} \frac{S_n}{f(n)} \geq 1.$$

(6) Prove that for any $n \in [\![\lambda^k, \lambda^{k+1}]\!]$ and $S_n > 0$ we have

$$\frac{S_n}{f(n)} \leq \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)} + \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)}.$$

(7) Prove that

$$\mathbb{P}(D_k) \underset{k \to \infty}{\sim} 2\mathbb{P}\left(X_1 \geq \frac{\alpha f(\lambda^k)}{\sqrt{\lambda^{k+1} - \lambda^k}} \right) \underset{k \to \infty}{\sim} \frac{c}{\sqrt{\log \lambda}} \left(\frac{1}{k} \right)^{\frac{\alpha^2}{\lambda - 1}}.$$

(8) Prove that for $\alpha^2 > \lambda - 1$, almost surely

$$\limsup_{n \to \infty} \frac{S_n}{f(n)} \leq \limsup_{n \to \infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} + \alpha.$$

(9) By choosing appropriate λ and α, prove that almost surely

$$\limsup_{n \to \infty} \frac{S_n}{f(n)} \leq 1.$$

(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?