Probability, homework 2, due September 20.

Exercise 1. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ a sequence of events in $\mathcal{A}$ which converges to A. Prove that

(i) $A \in \mathcal{A}$;
(ii) $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Exercise 2. Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbb{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbb{1}_{[2,\infty)}(x)$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 3. Let μ be the Lebesgue measure on $\mathbb{R}$. Build a sequence of functions $(f_n)_{n\geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \mathrm{d}\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n\geq 0}$ does not converge.

Exercise 4. Let X be a random variable in $\mathrm{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n\geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_N) \underset{n\to\infty}{\longrightarrow} 0$. Prove that $\mathbb{E}(X\mathbb{1}_{A_n}) \underset{n\to\infty}{\longrightarrow} 0$.

Exercise 5. Let $(d_n)_{n\geq 0}$ be a sequence in $(0,1)$, and $K_0 = [0,1]$. We define iteratively $(K_n)_{n\geq 0}$ in the following way. From K_n, which is the union of closed disjoint intervals, we define K_{n+1} by removing from each interval of K_n an open interval, centered at the middle of the previous one, with length d_n times the length of the previous one. Let $K = \cap_{n\geq 0} K_n$ (K is called a Cantor set).

(a) Prove that K is an uncountable compact set, with empty interior, and whose points are all accumulation points

(b) What is the Lebesgue measure of K?

Exercise 6. Let X be a nonnegative random variable. Prove that $\mathbb{E}(X) < +\infty$ if and only if $\sum_{n\in\mathbb{N}} \mathbb{P}(X \geq n) < \infty$.

Exercise 7. *Convergence in measure.* Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. and $(f_n)_{n\geq 1}, f : \Omega \to \mathbb{R}$ measurable (for the Borel σ-field on $\mathbb{R}$). We say that $(f_n)_{n\geq 1}$ converges in measure to f if for any $\varepsilon > 0$ we have

$$\mu(|f_n - f| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0.$$

(i) Show that $\int |f - f_n|\mathrm{d}\mu \to 0$ implies that f_n converges to f in measure. Is the reciprocal true?
(ii) Show that if $f_n \to f$ μ-almost surely, then $f_n \to f$ in measure. Is the reciprocal true?
(iii) Show that if $f_n \to f$ in measure, there exists a subsequence of $(f_n)_{n\geq 1}$ which converges μ-almost surely.
(iv) *(A stronger dominated convergence theorem)* We assume that $f_n \to f$ in measure and $|f_n| \leq g$ for some integrable $g : \Omega \to \mathbb{R}$, for any $n \geq 1$.
    (a) Show that $|f| \leq g$ μ-a.s.
    (b) Deduce that $\int |f_n - f|\mathrm{d}\mu \to 0$.

Exercise 8. Consider a probability space $(\Omega, \mathcal{A}, \mu)$ and $(A_n)_n$ a sequence in $\mathcal{A}$. Let $f : \Omega \to \mathbb{R}$ be measurable (for the Borel σ-field on $\mathbb{R}$) such that $\int_\Omega |\mathbb{1}_{A_n} - f|\mathrm{d}\mu \to 0$ as $n \to \infty$. Prove that there exists $A \in \mathcal{A}$ such that $f = \mathbb{1}_A$ μ-a.s., i.e. $\mu(f = \mathbb{1}_A) = 1$.