Probability, homework 10, due December 6.

Exercise 1. Consider a Markov chain X with state space $\mathbb{N}$ and transition matrix

$\pi(0,0) = r_0,\ \pi(0,1) = p_0$, and $\forall i \geq 1,\ \pi(i,i-1) = q_i,\ \pi(i,i) = r_i,\ \pi(i,i+1) = p_i$,

with $p_0, r_0 > 0$, $p_0 + r_0 = 1$ and for all $i \geq 1$, $p_i, q_i > 0$, $p_i + q_i + r_i = 1$. Prove that the chain is irreducible, aperiodic. Give a necessary and sufficient condition for the chain to have an invariant probability measure.

Exercise 2. Let $(G, \cdot)$ be a group, μ a probability measure on G and X the Markov chain such that $\pi(g, h \cdot g) = \mu(h)$. We call such a process X a random walk on G with jump kernel μ.

(i) Explain why the usual random walk on $\mathbb{Z}^d$ is such process. Same question for the usual random walk on $(\mathbb{Z}/n\mathbb{Z})^d$, $n \geq 1$.

(ii) Consider the following shuffling of a deck of $n \geq 2$ cards: pick two such distinct cards uniformly at random and exhange their positions in the deck. Show that this is also an example of a random walk on a group.

(iii) Let $\mathcal{H} = \{h_1 \cdot h_2 \cdot \dots \cdot h_n, \mu(h_i) > 0, 1 \leq i \leq n, n \in \mathbb{N}\}$. Discuss irreductibility of X depending on $\mathcal{H}$.

(iv) If X is irreducible on finite G, what are the invariant probability measures? What if G is not finite?

(v) Make some search to define a reversible Markov chain. In the context of this exercise, show that X is reversible if and only if $\mu(h) = \mu(h^{-1})$ for any $h \in G$.

(vi) Give an example of an irreducible random walk on a group which is not reversible.

Exercise 3. Let $(X_n)_{n\geq1}$ be independent such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Find sequences $(b_n)_{n\geq1}$, $(c_n)_{n\geq1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq1}$ is a $(\mathcal{F}_n)_{n\geq1}$-martingale.

b) Assume moreover that there is a real number λ such that $e^{\lambda X_i} \in \mathrm{L}^1$ for any $i \geq 1$. Find a sequence $(a_n^{(\lambda)})_{n\geq1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq1}$ is a $(\mathcal{F}_n)_{n\geq1}$-martingale.

Exercise 4. Let $(X_k)_{k\geq0}$ be i.i.d. random variables, $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ and $Y_m = \prod_{k=1}^m X_k$. Under which conditions is $(Y_m)_{m\geq1}$ a $(\mathcal{F}_m)_{m\geq1}$-submartingale, supermartingale, martingale?

Exercise 5. Let $(\mathcal{F}_n)_{n\geq0}$ be a filtration, $(X_n)_{n\geq0}$ a sequence of integrable random variables with $\mathbb{E}\left(X_n \mid \mathcal{F}_{n-1}\right) = 0$, and assume X_n is $\mathcal{F}_n$-measurable for every n. Let $S_n = \sum_{k=0}^n X_k$. Show that $(S_n)_{n\geq0}$ is a $(\mathcal{F}_n)_{n\geq0}$-martingale.

Exercise 6. Let $a > 0$ be fixed, $(X_i)_{i\geq1}$ be iid, $\mathbb{R}^d$-valued random variables, uniformly distributed on the ball $\mathrm{B}(0, a)$. Set $S_n = x + \sum_{i=1}^n X_i$.

(i) Let f be a superharmonic function. Show that $(f(S_n))_{n\geq1}$ defines a supermartingale.

(ii) Prove that if $d \leq 2$ any nonnegative superharmonic function is constant. Does this result remain true when $d \geq 3$?

In the following exercise you can use the following fact: If a martingale is a.s. bounded by a deterministic constant, it converges almost surely.

Exercise 7. Let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of random variables, and assume (Y_n) converges in distribution to a limiting Y. Also, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the sequence of independent random variables $X := (X_n)_{n\in\mathbb{N}^*}$ is defined, and we assume that the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ (i.e. $S_0 = 0$ and $S_n := \sum_{j=1}^n X_j$) converges in distribution. Set $(\mathcal{F}_n)$ the natural filtration of X and $\Phi_n(t) = \mathbb{E}(\exp(\mathrm{i}tS_n))$ for $t \in \mathbb{R}$.

- (i) Establish that $(\Phi_{Y_n}(\cdot))_{n\geq 1}$ converges uniformly on every compact, i.e. show that for any $a > 0$, $\max_{t\in[-a,a]} |\Phi_{Y_n}(t) - \Phi_Y(t)| \to 0$ as $n \to \infty$. Establish moreover that there exists $a > 0$ such that for any $n \geq 1$, $\min_{t\in[-a,a]} |\Phi_{Y_n}(t)| \geq 1/2$.
- (ii) Show that there exists $t_0 > 0$ such that if $t \in [-t_0, t_0]$, then $(\exp(\mathrm{i}tS_n)/\Phi_n(t))_{n\geq 0}$ is a $(\mathcal{F}_n)-$martingale (i.e. both its real and imaginary parts are martingales).
- (iii) Prove that we can choose $t_0 > 0$ such that for any $t \in [-t_0, t_0]$, $\lim_{n\to\infty} \exp(\mathrm{i}tS_n)$ exists $\mathbb{P}$-a.s.
- (iv) Set
$$C = \{(t, \omega) \in [-t_0, t_0] \times \Omega : \lim_{n\to\infty} \exp(\mathrm{i}tS_n(\omega)) \text{ exists}\}.$$
Prove that C is measurable, i.e. in the product of $\mathcal{B}([-t_0, t_0])$ with $\mathcal{F}$.
- (v) Establish that $\int_{-t_0}^{t_0} \mathbb{1}_C(t, \omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}t = 2t_0$.
- (vi) Prove that $\lim_{n\to\infty} S_n$ exists $\mathbb{P}$-a.s.