Probability, homework 8, due November 16.

Exercise 1. Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}, \mathcal{H}$ sub σ-fields of $\mathcal{F}$ such that $\sigma(\mathcal{G}, \mathcal{H}) = \mathcal{F}$. Find counterexamples to the following assertions:

(i) If $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ then X and Y are independent.
(ii) If $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}] = 0$ then $X = 0$.
(iii) If X and Y are independent then so are $\mathbb{E}[X \mid \mathcal{G}]$ and $\mathbb{E}[Y \mid \mathcal{G}]$.

Exercise 2. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Show that $|\mathbb{E}(Y \mid \mathcal{G})| \leq \mathbb{E}(|Y| \mid \mathcal{G})$ (almost surely).

Exercise 3. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-field of $\mathcal{G}$. Show that $\mathbb{E}(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(Y \mid \mathcal{H})$ (almost surely).

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequece of nonnegative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and $(\mathcal{F}_n)_{n\geq 0}$ a sequence of sub σ-fields of $\mathcal{F}$. Assume that $\mathbb{E}(X_n \mid \mathcal{F}_n)$ converges to 0 in probability.

(i) Show that X_n converges to 0 in probability.
(ii) Show that the reciprocal is wrong.

Exercise 5. Let $(X_n)_{n\geq 1}$ be independent such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Find sequences $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

b) Assume moreover that there is a real number λ such that $e^{\lambda X_i} \in \mathrm{L}^1$ for any $i \geq 1$. Find a sequence $(a_n^{(\lambda)})_{n\geq 1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

Exercise 6. Let $(X_k)_{k\geq 0}$ be i.i.d. random variables, $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ and $Y_m = \prod_{k=1}^m X_k$. Under which conditions is $(Y_m)_{m\geq 1}$ a $(\mathcal{F}_m)_{m\geq 1}$-submartingale, supermartingale, martingale?

Exercise 7. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, $(X_n)_{n\geq 0}$ a sequence of integrable random variables with $\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = 0$, and assume X_n is $\mathcal{F}_n$-measurable for every n. Let $S_n = \sum_{k=0}^n X_k$. Show that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

Exercise 8. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\mathcal{F}_T$ is a σ-field.

Exercise 9. Let S and T be stopping times for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\max(S, T)$ and $\min(S, T)$ are stopping times.

Exercise 10. Let $S \leq T$ be two stopping times and $A \in \mathscr{F}_S$. Define $U(\omega) = S(\omega)$ if $\omega \in A$, $U(\omega) = T(\omega)$ if $\omega \notin A$. prove that U is a stopping time.