Probability, homework 6, due October 26.

Exercise 1. Let X have distribution function $F(x) = e^{-e^{-x}}$. Justify that such a probability measure on $\mathbb{R}$ exists. Let $Y = F(X)$. Calculate $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Exercise 2. Assume $(\Omega, \mathcal{A}, \mathbb{P})$ is such that Ω is countable and $\mathcal{A} = 2^{\Omega}$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 3. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 4. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Exercise 5. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^{n} X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

Exercise 6. Calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Exercise 7. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 8. A sequence of random variables $(X_i)_{i\geq 1}$ is said to be completely convergent to X if for any $\varepsilon > 0$, we have $\sum_{i\geq 1} \mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that if the X_i's are independent then complete convergence implies almost sure convergence.

Exercise 9. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_\ell) = \mu$ for any ℓ, and a weak correlation in the following sense: $\mathrm{Cov}(X_k, X_\ell) \leq f(|k-\ell|)$ for all indexes k, ℓ, where the sequence $(f(m))_{m\geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1}\sum_{k=1}^{n} X_k)_{n\geq 1}$ converges to μ in L^2.

Exercise 10. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, on the same probability space, with law given by $\mathbb{P}(X_1 = (-1)^m m) = 1/(cm^2 \log m)$ for $m \geq 2$ (c is the normalization constant $c = \sum_{m\geq 2} 1/(m^2 \log m)$). Prove that $\mathbb{E}(|X_1|) = \infty$, but there exists a constant $\mu \notin \{\pm\infty\}$ such that $(n^{-1}\sum_{k=1}^{n} X_k)_{n\geq 1}$ converges to μ in probability. Does it converge almost surely, and in L^p?