**Probability, homework 5, due October 19.**

**Exercise 1**. Prove that if a sequence of real random variables $(X_n)$ converge in distribution to $X$, and $(Y_n)$ converges in distribution to a constant $c$, then $X_n+Y_n$ converges in distribution to $X+c$.

**Exercise 2.** Assume that $(X,Y)$ has joint density
$$ce^{-(1+x^2)(1+y^2)},$$
where $c$ is properly chosen. Are $X$ and $Y$ Gaussian random variables? Is $(X,Y)$ a Gaussian vector?

**Exercise 3**. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_j^2)=\sigma^2>0$. Let $S_n=X_1+\cdots+X_n$. Prove that
$$\lim_{n\to\infty}\mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right)=\sqrt{\frac{2}{\pi}}\sigma.$$

**Exercise 4**. Let the $X_\ell$'s be independent uniformly bounded real random variables. Let $\mu_\ell=\mathbb{E}(X_\ell)$, and $\sigma_\ell^2=\mathrm{Var}(X_\ell)$ satisfy $c_1<\sigma_\ell^2$ for some $c_1$ which does not depend on $\ell$. State and prove a central limit theorem for $\sum_{\ell=1}^n X_\ell$.

**Exercise 5**. Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables, with $X_i$ uniform on $[-i,i]$. Let $S_n=X_1+\cdots+X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.

**Long problem.** The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for $X_1,X_2\dots$ i.i.d. standard Gaussian random variables, denoting $S_n=X_1+\cdots+X_n$, we have
$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=1\right)=1 \tag{0.1}$$

(1) Prove that
$$\mathbb{P}(X_1>\lambda)\underset{\lambda\to\infty}{\sim}\frac{1}{\lambda\sqrt{2\pi}}e^{-\frac{\lambda^2}{2}}.$$
In the following questions we denote $f(n)=\sqrt{2n\log\log n}$, $\lambda>1$, $c,\alpha>0$, $A_k=\{S_{\lfloor\lambda^k\rfloor}\geq cf(\lambda^k)\}$, $C_k=\{S_{\lfloor\lambda^{k+1}\rfloor}-S_{\lfloor\lambda^k\rfloor}\geq cf(\lambda^{k+1}-\lambda^k)\}$ and $D_k=\{\sup_{n\in[\![\lambda^k,\lambda^{k+1}]\!]}\frac{S_n-S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)}\geq\alpha\}$.

(2) Prove that for any $c>1$ we have $\sum_{k\geq 1}\mathbb{P}(A_k)<\infty$ and
$$\limsup_{k\to\infty}\frac{S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)}\leq 1 \text{ a.s.}$$

(3) Prove that for any $c<1$ we have $\sum_{k\geq 1}\mathbb{P}(C_k)=\infty$ and
$$\mathbb{P}(C_k \text{ i.o.})=1.$$

(4) Let $\varepsilon>0$ and choose $c=1-\varepsilon/10$. Prove that almost surely the following inequality holds for infinitely many $k$:
$$\frac{S_{\lfloor\lambda^{k+1}\rfloor}}{f(\lambda^{k+1})}\geq c\frac{f(\lambda^{k+1}-\lambda^k)}{f(\lambda^{k+1})}-(1+\varepsilon)\frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) By choosing a large enough $\lambda$ in the previous inequality, prove that almost surely
$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \geq 1.$$
(6) Prove that for any $n \in [\![\lambda^k, \lambda^{k+1}]\!]$ we have
$$\frac{S_n}{f(n)} \leq \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)} + \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lfloor \lambda^k \rfloor)}.$$
(7) Prove that
$$\mathbb{P}(D_k) \underset{k\to\infty}{\sim} 2\mathbb{P}\left(X_1 \geq \frac{\alpha f(\lambda^k)}{\sqrt{\lambda^{k+1} - \lambda^k}}\right) \underset{k\to\infty}{\sim} \frac{c}{\sqrt{\log \lambda}} \left(\frac{1}{k}\right)^{\frac{\alpha^2}{\lambda - 1}}.$$
(8) Prove that for $\alpha^2 > \lambda - 1$, almost surely
$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq \limsup_{n\to\infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} + \alpha.$$
(9) By choosing appropriate $\lambda$ and $\alpha$, prove that almost surely
$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq 1.$$
(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?