Probability, homework 4 due October 5.

Exercise 1. Let X be a random variable with density $f_X(x) = (1-|x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1-\cos u)}{u^2}.$$

Exercise 2.

(1) Prove that $\hat{\mu}$ is real-valued if and only if μ is symmetric, i.e. $\mu(A) = \mu(-A)$ for any Borel set A

(2) If X and Y are i.i.d., prove that $X - Y$ has a symmetric distribution.

Exercise 3. Let X_λ be a real random variable, with Poisson distribution with parameter λ. Calculate the characteristic function of X_λ. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Exercise 4. Assume that the sequence of random variables $(X_n)_{n\geq 1}$ satisfies $\mathbb{E}\, X_n \to 1$ and $\mathbb{E}\, X_n^2 \to 1$. Prove that $(X_n)_{n\geq 1}$ converges in distribution. What is the limit?

Exercise 5. Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be real random variables, with X_n and Y_n independent for any $n \geq 1$, and assume that X_n converges in distribution to X and Y_n to Y. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

Exercise 6. Let X, Y be independent and assume that for some constant α we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that X and Y are both constant random variables.

Exercise 7. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution and let $M_n = \max(X_1, \ldots, X_n)$. Prove that $(nM_n^{-1})_{n\geq 1}$ converges in distribution and identify the limit.

Exercise 8 Let X, Y be i.i.d., with characteristic functions denoted φ_X, φ_Y, and suppose $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = 1$. Assume also that $X+Y$ and $X-Y$ are independent.

(1) Prove that
$$\varphi_X(2u) = (\varphi_X(u))^3 \varphi_X(-u)$$

(2) Prove that X is a standard Gaussian random variable.

Exercise 9. For any $d \geq 1$, we admit that there is only one probability measure μ on $\mathcal{S}_d$, (the $(d-1)$-th dimensional sphere embedded in $\mathbb{R}^d$) that is uniform, in the following sense: for any isometry $A \in \mathrm{O}(d)$ (the orthogonal group in $\mathbb{R}^d$), and any continuous function $f : \mathcal{S}_d \to \mathbb{R}$,

$$\int_{\mathcal{S}_d} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{S}_d} f(Ax)\mathrm{d}\mu(x).$$

Let $X = (X_1, \ldots, X_d)$ be a vector of independent centered and reduced Gaussian random variables.

a) Prove that the random variable $U = X/\|X\|_{L^2}$ is uniformly distributed on the sphere.

b) Prove that, as $d \to \infty$, the main part of the globe is concentrated close to the Equator, i.e. for any $\varepsilon > 0$,

$$\int_{x\in\mathcal{S}_d, |x_1|<\epsilon} \mathrm{d}\mu(x) \to 1.$$