

Probability, homework 10, due December 7.

Exercise 1. For fixed $p, q \in [0, 1]$, consider a Markov chain X with two states $\{1, 2\}$, with transition matrix

$$\pi = (\pi(i, j))_{1 \leq i, j \leq 2} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

- (i) For which p, q is the chain irreducible? Aperiodic (see Lemma 4.14 and Remark 4.10 in Varadhan)?
- (ii) What are the invariant probability measures of X ?
- (iii) Compute $\pi^{(n)}$, $n \geq 1$.
- (iv) When X is irreducible, for this invariant probability measure μ , calculate

$$d_1(n) := \frac{1}{2} (|\mathbb{P}_1(X_n = 1) - \mu(1)| + |\mathbb{P}_1(X_n = 2) - \mu(2)|)$$

$$d_2(n) := \frac{1}{2} (|\mathbb{P}_2(X_n = 1) - \mu(1)| + |\mathbb{P}_2(X_n = 2) - \mu(2)|)$$

where \mathbb{P}_x means the chain starts at x .

Exercise 2. Consider a Markov chain X with state space $\{0, 1, \dots, n\}$ and transition matrix

$$\pi(0, k) = \frac{1}{2^{k+1}}, \quad 0 \leq k \leq n-1, \quad \pi(0, n) = \frac{1}{2^n}$$

$$\pi(k, k-1) = 1, \quad 1 \leq k \leq n-1, \quad \pi(n, n) = \pi(n, n-1) = \frac{1}{2}.$$

- (i) Prove that the chain has a unique invariant probability measure μ and calculate it.
- (ii) Prove that for any $0 \leq x_0 \leq n-1$, $\pi^{(x_0+1)}(x_0, \cdot) = \mu$.
- (iii) Prove that for any $0 \leq x_0 \leq n$, $\pi^{(n)}(x_0, \cdot) = \mu$.
- (iv) For any $t \geq 1$, calculate

$$d(t) := \frac{1}{2} \sum_{x=0}^n \left| \pi^{(t)}(n, x) - \mu(x) \right|,$$

and plot $t \mapsto d(t)$.

Exercise 3. Consider a Markov chain X with state space \mathbb{N} and transition matrix

$$\pi(0, 0) = r_0, \quad \pi(0, 1) = p_0, \quad \text{and } \forall i \geq 1, \quad \pi(i, i-1) = q_i, \quad \pi(i, i) = r_i, \quad \pi(i, i+1) = p_i,$$

with $p_0, r_0 > 0$, $p_0 + r_0 = 1$ and for all $i \geq 1$, $p_i, q_i > 0$, $p_i + q_i + r_i = 1$. Prove that the chain is irreducible, aperiodic. Give a necessary and sufficient condition for the chain to have an invariant probability measure.

Exercise 4. Let (G, \cdot) be a group, μ a probability measure on G and X the Markov chain such that $\pi(g, h \cdot g) = \mu(h)$. We call such a process X a random walk on G with jump kernel μ .

- (i) Explain why the usual random walk on \mathbb{Z}^d is such process. Same question for the usual random walk on $(\mathbb{Z}/n\mathbb{Z})^d$, $n \geq 1$.
- (ii) Consider the following shuffling of a deck of $n \geq 2$ cards: pick two such distinct cards uniformly at random and exchange their positions in the deck. Show that this is also an example of a random walk on a group.

- (iii) Let $\mathcal{H} = \{h_1 \cdot h_2 \cdots h_n, \mu(h_i) > 0, 1 \leq i \leq n, n \in \mathbb{N}\}$. Discuss irreducibility of X depending on \mathcal{H} .
- (iv) If X is irreducible on finite G , what are the invariant probability measures? What if G is not finite?
- (v) Make some search to define a reversible Markov chain. In the context of this exercise, show that X is reversible if and only if $\mu(h) = \mu(h^{-1})$ for any $h \in G$.
- (vi) Give an example of an irreducible random walk on a group which is not reversible.