Exercise 1. Let $X_\lambda$ be a real random variable, with Poisson distribution with parameter $\lambda$. Calculate the characteristic function of $X_\lambda$. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Solution. We have $P(X_\lambda = k) = e^{-\lambda} \lambda^k / k!$, so that

$$E(e^{iuX_\lambda}) = \sum_k e^{iku} e^{-\lambda} \lambda^k / k! = e^{\lambda(e^{iu} - 1)}.$$

This implies, for fixed $u$ as $\lambda \to \infty$,

$$E(e^{iu(X_\lambda - \lambda)/\sqrt{\lambda}}) = e^{\lambda(e^{iu\sqrt{\lambda}} - 1)} e^{-iu\sqrt{\lambda}} = \exp\left(\lambda\left(iu - \frac{u^2}{2\lambda} + O(\lambda^{-3/2})\right) - iu\sqrt{\lambda}\right)$$

$$= e^{-\frac{u^2}{2} + o(1)} = E(e^{iuN(0,1)})/(1 + o(1)).$$

By Lévy’s theorem, this concludes the proof that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian.

Exercise 2. Assume a probability space $(\Omega, A, P)$ is such that $\Omega$ is countable and $A = 2^{\Omega}$. Prove that convergence in probability and convergence almost sure are the same.

Solution. We already know that convergence a.s. implies convergence in probability. Assume now that $X_n \to X$ in probability.

As $A = 2^{\Omega}$, the set $A$ of $\omega$’s such that $P(\{\omega\}) > 0$ is measurable. We need to prove that the measure of $\omega$’s such that $X_n(\omega) \to X(\omega)$ is 1, which is equivalent to $X_n(\omega) \to X(\omega)$ for any $\omega \in A$.

From convergence in probability, for any $\varepsilon > 0$ we have

$$\sum_{\omega \in A} P(\{\omega\}) 1_{|X_n(\omega) - X(\omega)| > \varepsilon} \to 0$$

as $N \to \infty$. In particular, for any $\omega \in A$, $1_{|X_n(\omega) - X(\omega)| > \varepsilon} \to 0$. As this is true for any $\varepsilon$, this means that for any $\omega \in A$ we have $X_n(\omega) \to X(\omega)$, which concludes the proof.

Exercise 3. Calculate

$$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Solution. Let $X_n$ be a Poisson random variable with parameter $n$. Then the sum of interest can exactly be interpreted as

$$P(X_n \leq n) = P\left(\frac{X_n - n}{\sqrt{n}} \leq 0\right).$$

From Exercise 1, we know this converges (as $n \to \infty$) to $P(N(0,1) \leq 0) = \frac{1}{2}$.

Exercise 4. Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. Bernoulli random variables, on the same probability space, with parameter 1/2 ($P(X_n = 0) = P(X_n = 1) = 1/2$), and let $\tau_n$ be the hitting time of level $n$ by the partial sums, i.e. $\tau_n = \inf\{k \mid$
\[ \sum_{\ell=1}^{k} X_{\ell} = n \}. \] Show that \( n^{-1} \tau_n \) converges to 2 almost surely.

**Solution.** We need to prove two claims:

\[
P(n^{-1} \tau_n > 2 + \varepsilon) \to 0, \quad (0.1)
\]

\[
P(n^{-1} \tau_n < 2 - \varepsilon) \to 0. \quad (0.2)
\]

Let \( S_k = \sum_{\ell=1}^{k} X_{\ell} \). Then the LHS of (0.1) is bounded by

\[
P \left( S_{(2+\varepsilon)n} < n \right) = P \left( \frac{S_{(2+\varepsilon)n} - (1 + \varepsilon/2)n}{(2+\varepsilon)n} < -\frac{\varepsilon}{2(2+\varepsilon)} \right)
\]

By the weak law of large numbers, the above RHS converges to 0 as \( n \to \infty \).

Concerning (0.2), the LHS can be evaluated by this union bound:

\[
\sum_{k \leq (2-\varepsilon)n} P(S_k \geq n) = \sum_{k \leq (2-\varepsilon)n} P \left( \frac{S_k - k}{\sqrt{k}} \geq \frac{n - k}{\sqrt{k}} \right)
\]

By Markov’s inequality and optimization in \( u > 0 \), we have for any \( v > 0 \)

\[
P \left( \frac{S_k - k}{\sqrt{k}} \geq v \right) \leq E \left( e^{u \left( \frac{S_k - k}{\sqrt{k}} \right)} \right) e^{-uv} \leq e^{-cv^2}
\]

for some universal \( c > 0 \). We therefore have bounded the left hand side of (0.2) by

\[
\sum_{k \leq (2-\varepsilon)n} \exp \left( -c \left( \frac{n - k}{\sqrt{k}} \right)^2 \right) \leq (2-\varepsilon)n \exp \left( -c \left( \frac{n - (2-\varepsilon)n}{\sqrt{(2-\varepsilon)n}} \right)^2 \right) \to 0.
\]

**Exercise 5.** Let \( \alpha > 0 \) and, given \((\Omega, \mathcal{A}, P)\), let \((X_n, n \geq 1)\) be a sequence of independent real random variables with law \( P(X_n = 1) = \frac{1}{n^\alpha} \) and \( P(X_n = 0) = 1 - \frac{1}{n^\alpha} \). Prove that \( X_n \to 0 \) in \( L^1 \), but that almost surely

\[
\lim \sup_{n \to \infty} X_n = \begin{cases} 1 & \text{if } \alpha \leq 1 \smallskip \\ 0 & \text{if } \alpha > 1 \end{cases}.
\]

**Solution.** For convergence to 0 in \( L^1 \), we just write

\[
E(|X_n|) = n^{-\alpha} \to 0.
\]

If \( \alpha > 1 \), then \( \sum P(X_n = 1) < \infty \) so by Borel Cantelli \( P(X_n = 1 \ i.o.) = 0 \), so \( P(\lim \sup_{n \to \infty} X_n = 0) = 1 \).

If \( \alpha \leq 1 \), the reverse direction (which requires independent \( X_n \)'s) of Borel Cantelli also applies in a similar way.

**Exercise 6.** A sequence of random variables \((X_i)_{i \geq 1}\) is said to be completely convergent to \( X \) if for any \( \varepsilon > 0 \), we have \( \sum_{i \geq 1} P(|X_i - X| > \varepsilon) < \infty \). Prove that if the \( X_i \)'s are independent then complete convergence implies almost sure convergence.

**Solution.** By Borel Cantelli, under the assumption of complete convergence, for any \( \varepsilon > 0 \) we have

\[
P(|X_i - X| > \varepsilon \ i.o.) = 0.
\]

This implies

\[
P(\lim \sup(X_i - X) > \varepsilon) = P(\lim \inf(X_i - X) < -\varepsilon) = 0.
\]
By monotonicity of the sets, taking \( \varepsilon \to 0 \) in the above convergence we get

\[
\mathbb{P}(\limsup (X_i - X) > 0) = \mathbb{P}(\liminf (X_i - X) < 0) = 0,
\]

so that \( \mathbb{P}(\lim (X_i - X) - 0) = 1 \).

Note that we have not used the independence of the \( X_i \)'s. It could be used to establish the reciprocal.

**Exercise 7.** Let \( X, Y \) be independent and assume that for some constant \( \alpha \) we have \( \mathbb{P}(X + Y = \alpha) = 1 \). Prove that \( X \) and \( Y \) are both constant random variables.

**Solution.** Assume the contrary, i.e. e.g. the exists \([a, b] \) and \([c, d] \) with \( b < c \) such that \( \mathbb{P}(X \in [a, b]) > 0, \mathbb{P}(X \in [c, d]) > 0 \).

Let \([x, y]\) be such that \( y - x < c - b \) and \( \mathbb{P}(Y \in [x, y]) > 0 \).

Then we have

\[
\mathbb{P}(X + Y > c + x) \geq \mathbb{P}(X > c, Y > x) > 0
\]

and

\[
\mathbb{P}(X + Y < b + y) \geq \mathbb{P}(X < b, Y < y) > 0.
\]

The two equations above together with \( c + x > b + y \) contradict the fact that \( X + Y \) has distribution a Dirac mass.

**Exercise 8.** Let \( (X_i)_{i \geq 1} \) be a sequence of i.i.d. random variables with mean 0 and finite variance \( \mathbb{E}(X_i^2) = \sigma^2 > 0 \). Let \( S_n = X_1 + \cdots + X_n \). Prove that

\[
\lim_{n \to \infty} \mathbb{E}\left( \frac{|S_n|}{\sqrt{n}} \right) = \sqrt{\frac{2}{\pi}} \sigma.
\]

**Solution.** By the central limit theorem, for any continuous bounded \( f \) we have

\[
\mathbb{E} f \left( \frac{S_n}{\sqrt{n}} \right) \to \mathbb{E} f (\mathcal{N}(0, 1))
\]

If we could pick \( f(x) = |x| \), this would conclude the proof by direct calculation of the RHS. But \( f(x) = |x| \) is not bounded. Therefore, choose first \( f_M(x) = \min(|x|, M) \) and \( \varepsilon > 0 \) an arbitrary small fixed constant.

First, we have

\[
|\mathbb{E}(f_M(\mathcal{N}(0, 1))) - \mathbb{E}(\mathcal{N}(0, 1))| \leq 2 \int_M^\infty xe^{-x^2/2}/\sqrt{2\pi},
\]

so that we can find \( M \) large enough so that the above difference is at most \( \varepsilon \).

Moreover,

\[
|\mathbb{E}(f_M(S_n/\sqrt{n}) - |S_n/\sqrt{n}|) | \leq \mathbb{E}(|S_n/\sqrt{n}|) \mathbb{1}_{|S_n/\sqrt{n}| > M} \\
\leq \mathbb{E}(|S_n/\sqrt{n}|)^{1/2} \mathbb{P}(|S_n/\sqrt{n}| > M)^{1/2} = \sigma^2 \mathbb{P}(|S_n/\sqrt{n}| > M)^{1/2},
\]

by Cauchy-Schwarz. From the central limit theorem, we can find \( M, N_0 \) large enough such that the above RHS is smaller than \( \varepsilon \) for any \( N > N_0 \).

The above three displayed equations (the first one applied to \( f_M \), the next two being error terms smaller than \( \varepsilon \)) allow to conclude.

**Exercise 9.** Let \( (X_i)_{i \geq 1} \) be a sequence of independent random variables, with \( X_i \) uniform on \([-i,i] \). Let \( S_n = X_1 + \cdots + X_n \). Prove that \( S_n/n^{3/2} \) converges in distribution and describe the limit.
Solution. Let’s calculate the characteristic function ($Y_k$’s denote independent uniform random variables on $[-1, 1]$):

$$
E(\exp(iun^{-3/2})) = \prod_{k=1}^{n} E(\exp(iun^{-3/2}X_k)) = \prod_{k=1}^{n} E(\exp(iun^{-3/2}kY_k)) = \prod_{k=1}^{n} \frac{\sin(un^{-3/2}k)}{un^{-3/2}k}
$$

$$
= \prod_{k=1}^{n} \left(1 - \frac{(un^{-3/2}k)^3}{6un^{-3/2}k} + O((n^{-3/2}k)^4)\right) = \prod_{k=1}^{n} \exp \left(-\frac{(un^{-3/2}k)^2}{6} + O((n^{-3/2}k)^4)\right)
$$

As $\sum_{k=1}^{n} k^2 = n(n + 1)(2n + 1)/6 \sim n^3/3$, we obtain

$$
E(\exp(iun^{-3/2})) \to e^{-\frac{u^2}{3}},
$$

in other words $S_n n^{-3/2}$ converges in distribution to a Gaussian random variable with mean zero and variance $3^2$. 