Exercise 1. Let $X_\lambda$ be a real random variable, with Poisson distribution with parameter $\lambda$. Calculate the characteristic function of $X_\lambda$. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Exercise 2. Assume a probability space $(\Omega, A, P)$ is such that $\Omega$ is countable and $A = 2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 3. Calculate
$$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Exercise 4. Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. Bernoulli random variables, on the same probability space, with parameter $1/2$ ($P(X_n = 0) = P(X_n = 1) = 1/2$), and let $\tau_n$ be the hitting time of level $n$ by the partial sums, i.e. $\tau_n = \inf\{k \mid \sum_{\ell=1}^{k} X_\ell = n\}$. Show that $n^{-1}\tau_n$ converges to 2 almost surely.

Exercise 5. Let $\alpha > 0$ and, given $(\Omega, A, P)$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $P(X_n = 1) = \frac{1}{n^\alpha}$ and $P(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $L^1$, but that almost surely
$$\limsup_{n \to \infty} X_n = \begin{cases} 1 & \text{if } \alpha \leq 1 \\ 0 & \text{if } \alpha > 1 \end{cases}.$$

Exercise 6. A sequence of random variables $(X_i)_{i \geq 1}$ is said to be completely convergent to $X$ if for any $\varepsilon > 0$, we have $\sum_{i \geq 1} P(|X_i - X| > \varepsilon) < \infty$. Prove that if the $X_i$'s are independent then complete convergence implies almost sure convergence.

Exercise 7. Let $X, Y$ be independent and assume that for some constant $\alpha$ we have $P(X + Y = \alpha) = 1$. Prove that $X$ and $Y$ are both constant random variables.

Exercise 8. Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_i^2) = \sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$. Prove that
$$\lim_{n \to \infty} \mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right) = \sqrt{\frac{2}{\pi}} \sigma.$$

Exercise 9. Let $(X_i)_{i \geq 1}$ be a sequence of independent random variables, with $X_i$ uniform on $[-i, i]$. Let $S_n = X_1 + \cdots + X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.