Probability, homework 6.

Exercise 1. In the (O, x, y) plane, a random ray emerges from a light source at point $(-1, 0)$, towards the (O, y) axis. The angle with the (O, x) axis is uniform on $(-\frac{\pi}{2}, \frac{\pi}{2})$. What is the distribution of the impact point with the (O, y) axis?

Exercise 2. Let X be uniform on $(-\pi, \pi)$ and $Y = \sin(X)$. Show that the density of Y is

$$\frac{1}{\pi\sqrt{1-y^2}} \mathbb{1}_{[-1,1]}(y).$$

Exercise 3. Let (X, Y) be uniform on the unit ball, i.e. it has density

$$f_{(X,Y)}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if} \quad x^2 + y^2 \leq 1, \\ 0 & \text{if} \quad x^2 + y^2 > 1. \end{cases}$$

Find the density of $\sqrt{X^2 + Y^2}$.

Exercise 4. Let (X, Y) have density $\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$. What is the density of X/Y?

Exercise 5. Let $(X_i)_{i \geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 6. Let f be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \cdots + x_n}{n}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Exercise 7. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0, 1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^{n} X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.