Probability, homework 5, due March 4.

Exercise 1. Build a sequence of functions \( (f_n)_{n \geq 0} \), \( 0 \leq f_n \leq 1 \), such that \( \int f_n \, d\mu \rightarrow 0 \) but for any \( x \in \mathbb{R} \), \( (f_n(x))_{n \geq 0} \) does not converge.

Solution. Define \( f_n(x) = 1_{x \in [f(n), g(n)]} \) where, for \( n \in [2^p, 2^{p+1}) \), \( f(n) = (n - 2^p - 2^{p-1})/p \), \( g(n) = f(n) + 1/p \). Clearly \( \int f_n = 1/p \rightarrow 0 \) and for any \( x \) we have \( f_n(x) = 1 \), and \( 0 \) i.o.

Exercise 2. Let \( X \) be a random variable in \( L^1(\Omega, \mathcal{A}, \mathbb{P}) \). Let \( (A_n)_{n \geq 0} \) be a sequence of events in \( \mathcal{A} \) such that \( \mathbb{P}(A_N) \rightarrow 0 \). Prove that \( \mathbb{E}(X1_{A_n}) \rightarrow 0 \).

Solution. For any \( C > 0 \) we have
\[
|\mathbb{E}(X1_{A_n})| \leq \mathbb{E}(|X|1_{|X|>C}) + C\mathbb{P}(A_n).
\]
Let \( \varepsilon > 0 \). By monotone convergence, there exists \( C > 0 \) such that \( \mathbb{E}(|X|1_{|X|>C}) < \varepsilon \). For this \( C \), there exists a \( n_0 \) such that for any \( n > n_0 \) we have \( \mathbb{P}(A_n) < \varepsilon/C \). Thus we have proved that for \( n > n_0 \), \( |\mathbb{E}(X1_{A_n})| < 2\varepsilon \), which concludes the proof.

Exercise 3. Let \( X, Y \) be random variables such that \( X, Y \) ad \( XY \) are in \( L^1 \). Assume \( \mathbb{E}(XY) = \mathbb{E}(X) \mathbb{E}(Y) \). By giving an example, prove that \( X \) and \( Y \) are not necessarily independent.

Solution. Let \( G \) be a standard Gaussian random variable and \( B \) an independent Bernoulli \( \pm 1 \) with parameter \( 1/2 \). Pick \( X = G, Y = BG \). Then all expectations are 0 and in particular \( \mathbb{E}(XY) = \mathbb{E}(X) \mathbb{E}(Y) \). However \( X \) and \( Y \) are not independent because \( \mathbb{P}(X > 0, Y > 0) = 0 \neq \mathbb{P}(X > 0)\mathbb{P}(Y > 0) \).

Exercise 4. Let \( X, Y \) be in \( L^1 \). Prove that, if \( X \) and \( Y \) are independent, \( XY \in L^1 \). Show this is not true in general (i.e. if \( X \) and \( Y \) are not independent).

Solution. Let \( C > 0 \). As \( X \) and \( Y \) are independent and \( x \mapsto |x|1_{|x|<C} \) is bounded, we have
\[
\mathbb{E}(|X|1_{|X|<C}|Y|1_{|Y|<C}) = \mathbb{E}(|X|1_{|X|<C})\mathbb{E}(|Y|1_{|Y|<C}).
\]
Take the limit \( X \rightarrow \infty \) on the right hand side. By monotone convergence, this limit is finite (and equal to \( \mathbb{E}(|X|)\mathbb{E}(|Y|) \)). Hence the left hand side (which also converges to \( \mathbb{E}(|XY|) \) has a finite limit, i.e. \( \mathbb{E}(|XY|) < \infty \).

Exercise 5. A monkey tries to do some typesetting, by successively pushing one of the 84 keyboard buttons, forever, each button being chosen uniformly and independently of the others. Prove that almost surely, after some time, he will type the exact sequence of letters in James Joyce’s Ulysses.
Solution. Let \( n \) be the total number of letters in this novel. Let \( A_k \) be the event that the sequence of characters typed between pushes \( kn \) and \( (k+1)n - 1 \) coincides with this novel. Then there exists \( \varepsilon > 0 \) such that \( \mathbb{P}(A_k) > \varepsilon \) (more precisely \( \varepsilon = \ell^{-n} \) where \( \ell \) is the number of buttons). Hence by independence of the \( A_k \)'s we have \( \mathbb{P}(\cap_{k \geq 1} A_k) = 0 \), i.e. \( \mathbb{P}(\cup_{k \geq 1} A_k) = 1 \), as desired.

Exercise 6. Let \((S_n)_{n \geq 0}\) be a random walk, and denote \( X_n = S_n - S_{n-1} \) for \( i \geq 1 \).

a) Prove that for any \( A > 0 \), \( \mathbb{P} \left( \limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} > A \right) > 0 \).

b) Read about the tail \( \sigma \)-algebra and Kolmogorov’s 0-1 law in the courses’ book.

c) Prove that \( \{ \limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} > A \} \in \cap_{n \geq 1} \sigma(X_i, i \geq n) \).

d) Deduce that \( \mathbb{P} \left( \limsup_{n \to \infty} \frac{S_n}{\sqrt{n}} = +\infty \right) = 1 \).

Solution.

a) We have, by convergence of events,

\[
\mathbb{P}(\cap_{n \geq 1} \cup_{m \geq n} \{ S_m/\sqrt{m} > A \}) = \lim_{n \to \infty} \mathbb{P}(\cup_{m \geq n} \{ S_m/\sqrt{m} > A \}) \\
\geq \lim_{n \to \infty} \mathbb{P}(S_n/\sqrt{n} > A) \\
= \int_A e^{-x^2/2} \frac{1}{\sqrt{2\pi}} \, dx
\]

where the last equation is the central limit theorem as proved by de Moivre.

c) For any fixed \( n \), we have the equality of events

\[
\{ \limsup_{m \to \infty} \frac{S_m}{\sqrt{m}} > A \} = \{ \limsup_{m \to \infty} \frac{\sum_{i=n}^{m} X_i}{\sqrt{m}} > A \}
\]

and the above right hand side is clearly in \( \sigma(X_i, i \geq n) \).

d) By c) we can apply the 0-1 law, and the probability of the event of interest is 0 or 1. From a) it cannot be 0, so it is 1.