Probability, homework 4, due February 26.

Exercise 1. Let $(A_n)_{n\geq 0}$ be a set of pairwise disjoint events and $\mathbb{P}$ a probability. Show that $\lim_{n\to\infty}\mathbb{P}(A_n)=0$.

Exercise 2. Suppose a distribution function F is given by

$$F(x)=\frac{1}{4}\mathbb{1}_{[0,\infty)}(x)+\frac{1}{2}\mathbb{1}_{[1,\infty)}(x)+\frac{1}{4}\mathbb{1}_{[2,\infty)}(x)$$

What is the probability of the following events, $(-1/2,1/2)$, $(-1/2,3/2)$, $(2/3,5/2)$, $(3,\infty)$?

Exercise 3. Let X be random variable on a countable probability space. Suppose that $\mathbb{E}(|X|)=0$. Prove that $\mathbb{P}(X=0)=1$. Is it true, in general, that for any $\omega\in\Omega$ we have $X(\omega)=0$?

Exercise 4. Let $\mathbb{P}$ be a probability measure on Ω, endowed with a σ-algebra $\mathscr{A}$.

(i) What is the meaning of the following events, where all A_n's are elements of $\mathscr{A}$?

$$\liminf_{n\to\infty} A_n=\bigcup_{n\geq 1}\bigcap_{k\geq n}A_k,\quad \limsup_{n\to\infty} A_n=\bigcap_{n\geq 1}\bigcup_{k\geq n}A_k.$$

(ii) In the special case $\Omega=\mathbb{R}$ and $\mathcal{A}$ is its Borel σ-algebra, for any $p\geq 1$, let

$$A_{2p}=\left[-1,2+\frac{1}{2p}\right),\quad A_{2p+1}=\left(-2-\frac{1}{2p+1},1\right].$$

What are $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$?

(iii) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right)\leq\liminf_{n\to\infty}\mathbb{P}\left(A_n\right),\mathbb{P}\left(\limsup_{n\to\infty} A_n\right)\geq\limsup_{n\to\infty}\mathbb{P}\left(A_n\right).$$

Exercise 5. Let $c>0$ and X be a real random variable such that for any $\lambda\in\mathbb{R}$

$$\mathbb{E}\left(e^{\lambda X}\right)\leq e^{c\frac{\lambda^2}{4}}.$$

Prove that, for any $\delta>0$,

$$\mathbb{P}\left(|X|>\delta\right)\leq 2e^{-\frac{\delta^2}{c}}.$$

Exercise 6. You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1-p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k,\dots,2^{k+1}-1$. Prove that $\mathbb{P}(A_k \text{ i.o.})=1$ if $p\geq 1/2$, 0 otherwise.

Here, i.o. stands for "infinitely often", and A_k i.o. is the event $\cap_{n\geq 1}\cup_{m\geq n}A_m$.

Exercise 7. Let $\epsilon>0$ and X be uniformly distributed on $[0,1]$. Prove that, almost surely (i.e. the following event has probability 1), there exists only a finite number of rationals $\frac{p}{q}$, with $p\wedge q=1$, such that

$$\left|X-\frac{p}{q}\right|<\frac{1}{q^{2+\epsilon}}.$$