Exercise 1. Let \((A_n)_{n \geq 0}\) be a set of pairwise disjoint events and \(P\) a probability. Show that \(\lim_{n \to \infty} P(A_n) = 0\).

Exercise 2. Suppose a distribution function \(F\) is given by
\[
F(x) = \frac{1}{4} \mathbb{1}_{[0, \infty)}(x) + \frac{1}{2} \mathbb{1}_{[1, \infty)}(x) + \frac{1}{4} \mathbb{1}_{[2, \infty)}(x)
\]
What is the probability of the following events, \((-1/2, 1/2), (-1/2, 3/2), (2/3, 5/2), (3, \infty)\)?

Exercise 3. Let \(X\) be a random variable on a countable probability space. Suppose that \(\mathbb{E}(|X|) = 0\). Prove that \(P(X = 0) = 1\). Is it true, in general, that for any \(\omega \in \Omega\) we have \(X(\omega) = 0\)?

Exercise 4. Let \(\mathbb{P}\) be a probability measure on \(\Omega\), endowed with a \(\sigma\)-algebra \(\mathcal{A}\).
(i) What is the meaning of the following events, where all \(A_n\)'s are elements of \(\mathcal{A}\)?
\[
\liminf_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k, \quad \limsup_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k.
\]
(ii) In the special case \(\Omega = \mathbb{R}\) and \(\mathcal{A}\) is its Borel \(\sigma\)-algebra, for any \(p \geq 1\), let
\[
A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].
\]
What are \(\liminf_{n \to \infty} A_n\) and \(\limsup_{n \to \infty} A_n\)?
(iii) Prove that the following always holds:
\[
\mathbb{P} \left( \liminf_{n \to \infty} A_n \right) \leq \liminf_{n \to \infty} \mathbb{P}(A_n), \quad \mathbb{P} \left( \limsup_{n \to \infty} A_n \right) \geq \limsup_{n \to \infty} \mathbb{P}(A_n).
\]

Exercise 5. Let \(c > 0\) and \(X\) be a real random variable such that for any \(\lambda \in \mathbb{R}\)
\[
\mathbb{E}(e^{\lambda X}) \leq e^{c \lambda^2 / 2}.
\]
Prove that, for any \(\delta > 0\),
\[
\mathbb{P}(|X| > \delta) \leq 2e^{-\frac{\delta^2}{2c}}.
\]

Exercise 6. You toss a coin repeatedly and independently. The probability to get a head is \(p\), a tail is \(1-p\). Let \(A_k\) be the following event: \(k\) or more consecutive heads occur amongst the tosses numbered \(2^k, \ldots, 2^{k+1} - 1\). Prove that \(\mathbb{P}(A_k \text{ i.o.}) = 1\) if \(p \geq 1/2\), \(0\) otherwise.
Here, i.o. stands for “infinitely often”, and \(A_k\) i.o. is the event \(\bigcap_{n \geq 1} \bigcup_{m \geq n} A_m\).

Exercise 7. Let \(\epsilon > 0\) and \(X\) be uniformly distributed on \([0, 1]\). Prove that, almost surely (i.e. the following event has probability 1), there exists only a finite number of rationals \(p/q\), with \(p \wedge q = 1\), such that
\[
\left|X - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.
\]