Probability, homework 3, due February 19th

Exercise 1. The probability that a male driver makes an insurance claim in any given year is 0.3, while the probability that a female driver makes an insurance claim in any given year is 0.2. Furthermore, claims by the same driver in successive years are independent events. We assume equal numbers of male and female drivers.

What is the probability that a randomly chosen driver makes a claim in the first year (event A)? What is the probability that a randomly chosen driver makes a claim in the first and second years (event B)?

What is $\mathbb{P}(B \mid A)$, the probability that a randomly chosen driver makes a claim in the second year, conditionally to the fact that he/she made one on the first year? How can you explain that it is different from $\mathbb{P}(A)$ although claims in successive years are independent? If you are the head of an insurance company and want one more client, would you prefer one who had a claim the previous year or the contrary?

Exercise 2. Let X be a geometric random variable. Prove the following memoryless property: for $i, j > 0$,

$$\mathbb{P}(X > i + j \mid X \geq i) = \mathbb{P}(X > j).$$

Exercise 3. Suppose that Ω is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have finite complement. Prove that $\mathcal{A}$ is not a σ-algebra.

Exercise 4. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ (resp. $(B_n)_{n\geq 1}$) be a sequence of events in $\mathcal{A}$ which converges to A (resp. B). Assume that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0$ for all n. Show that

(i) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B) = \mathbb{P}(A \mid B)$;
(ii) $\lim_{n\to\infty} \mathbb{P}(A \mid B_n) = \mathbb{P}(A \mid B)$;
(iii) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B_n) = \mathbb{P}(A \mid B)$.

Exercise 5. Let A, B, C be three mutually independent events and $\mathbb{P}(B \cap C) \neq 0$. Prove that $\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A)$.

Exercise 6. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: consider the event E_n that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $(n-1)$ rolls.

Exercise 7. Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. For $a, b \in \mathbb{Z}$, let $T_a = \inf\{n \geq 0 : s_n = a\}$ and $T_{a,b} = \inf\{n \geq 0 : s_n = a \textit{ or } s_n = b\}$. For $x \in \mathbb{Z}$, let $\omega(x) = \mathbb{P}(s_{T_{a,b}} = b \mid s_0 = x)$.

Prove that for $a < x < b$, $\omega(x) = \frac{1}{2}(\omega(x+1) + \omega(x-1))$, provided we define $\omega(a) = 0$ and $\omega(b) = 1$. Conclude that

$$\omega(x) = \frac{x-a}{b-a}.$$

Prove that $\mathbb{P}(T_b < \infty) = 1$.