Probability, homework 2, due February 12.

Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha\in A}$ be an arbitrary family of σ-algebras defined on an abstract space Ω. Show that $\cap_{\alpha\in A}\mathcal{G}_\alpha$ is also a σ-algebra.

Exercise 2. Let $\mathcal{A}$ be a σ-algebra. Prove that if, for all $n\in\mathbb{N}$, $A_n\in\mathcal{A}$, then $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are in $\mathcal{A}$ (these limiting events are defined in Jacod-Protter).

Exercise 3. Prove the Bonferroni inequalities: if $A_i\in\mathcal{A}$ is a sequence of events, then

(i) $\mathbb{P}\left(\cup_{i=1}^n A_i\right)\geq\sum_{i=1}^n\mathbb{P}(A_i)-\sum_{i<j}\mathbb{P}(A_i\cap A_j)$,
(ii) $\mathbb{P}\left(\cup_{i=1}^n A_i\right)\leq\sum_{i=1}^n\mathbb{P}(A_i)-\sum_{i<j}\mathbb{P}(A_i\cap A_j)+\sum_{i<j<k}\mathbb{P}(A_i\cap A_j\cap A_k)$.

Exercise 4. Prove whether the following sets are countable or not.

(i) All intervals in $\mathbb{R}$ with rational endpoints.
(ii) All circles in the plane with rational radii and centers on the diagonal $x=y$.
(iii) All sequences of integers whose terms are either 0 or 1.

Exercise 5. Let $\varnothing\subsetneq A\subsetneq B\subsetneq\Omega$ (these are strict inclusions). What is the σ-algebra generated by $\{A,B\}$?

Exercise 6. Let $(s_n)_{n\geq 0}$ be a random walk. For $a\in\mathbb{Z}^*$, let $T_a=\inf\{n\geq 0: s_n=a\}$. Prove that $\mathbb{E}(T_a)=\infty$.