Exercise 1. Let $X_i, i \geq 1$, be iid random variables, $X_i \geq 0, E(X_i) = 1$. Prove that if $Y_n = \prod_{k=i}^n X_k$, $\mathcal{F}_n = \sigma(X_k, k \leq n)$, $(Y_n)_{n \geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Prove that if $\mathbb{P}(X_1 = 1) < 1$, $Y_n$ converges to 0 almost surely.

Solution. We have

$$E(|Y_n|) = E\left(\prod_{i=1}^n X_i\right) = E(X_1)^n < \infty.$$ 

Moreover, $Y_n$ is a measurable function of $X_1, \ldots, X_n$, so it is $\mathcal{F}_n$-measurable. In addition, for any $m < n$ we have

$$E(Y_n | \mathcal{F}_m) = E(X_1 \ldots X_m X_{m+1} \ldots X_n | \mathcal{F}_m) = X_1 \ldots X_m E(X_{m+1} \ldots X_n | \mathcal{F}_m) = Y_m E(X_{m+1} \ldots X_n) = Y_m,$$

where for the second equality we used that $X_1 \ldots X_m$ is $\mathcal{F}_m$-measurable, and the third equality relies on independence of $X_{m+1} \ldots X_n$ and $\mathcal{F}_m$.

For the convergence to 0, we consider two cases. First, if $\mathbb{P}(X_1 = 0) > 0$, then $X_k = 0$ i.o. with probability 1, and in particular $Y_n \to 0$.

Second, assume $\mathbb{P}(X_1 = 0) = 0$. By Jensen’s inequality, we have $E \log X_1 \leq \log E X_1 = 0$. As log is strictly concave, equality occurs if and only if the distribution of $X_1$ is a Dirac mass (at 1, necessarily). As it is not, we have $E \log X_1 < 0$ (note that this expectation may be $-\infty$).

Therefore there exists an $\varepsilon \in [0, 1]$ such that, writing $Z_k = (\log X_k) 1_{X_k > \varepsilon}$, we have $E(Z_k) < 0$ (this relies on monotone convergence as $\varepsilon \to 0$ and $\mathbb{P}(X_1 = 0) = 0$). As the $Z_k$’s are iid and in $L^1$ (bounded below for the negative part, and integrable above as seen by Jensen) we can apply the strong law of large numbers: $(\sum_{k=1}^n Z_k)/n \to -c$ for some $c > 0$, and in particular

$$e^{\sum_{k=1}^n Z_k} \to 0 \text{ a.s.}.$$ 

As $X_k \leq e^{Z_k}$, this concludes the proof.

Exercise 2. Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration, $(X_n)_{n \geq 0}$ a sequence of integrable random variables with $E(X_n | \mathcal{F}_{n-1}) = 0$, and assume $X_n$ is $\mathcal{F}_n$-measurable for every $n$. Let $S_n = \sum_{k=0}^n X_k$. Show that $(S_n)_{n \geq 0}$ is a $(\mathcal{F}_n)_{n \geq 0}$-martingale.

Solution. Integrability and $\mathcal{F}_n$-measurability are part of the assumptions.

Assume we can prove

$$E(S_{n+1} | \mathcal{F}_n) = S_n. \quad (0.1)$$

This would be enough for the martingale property by an immediate induction: Assuming $E(S_{n+k} | \mathcal{F}_n) = S_n$ we have

$$E(S_{n+k+1} | \mathcal{F}_n) = E(E(S_{n+k+1} | \mathcal{F}_{n+k}) | \mathcal{F}_n) = E(S_{n+k} | \mathcal{F}_n) = S_n,$$

where the first equality relies on $\mathcal{F}_n \subset \mathcal{F}_{n+k}$ and the second equality is (0.1).
Therefore we just need to prove (0.1). It is elementary:
\[ E(S_{n+1} \mid \mathcal{F}_n) = E(X_{n+1} + X_n + \cdots + X_0 \mid \mathcal{F}_n) = 0 + X_n + \cdots + X_0 = S_n. \]

**Exercise 3.** Let \( T \) be a stopping time for a filtration \( (\mathcal{F}_n)_{n \geq 1} \). Prove that \( \mathcal{F}_T \) is a \( \sigma \)-algebra.

**Solution.** We first remind the definition of \( \mathcal{F}_T \):
\[
\mathcal{F}_T = \{ A \in \mathcal{F} \mid \forall n \in \mathbb{N}, A \cap \{ T \leq n \} \in \mathcal{F}_n \}.
\]

We need to prove:

(i) \( \Omega \in \mathcal{F}_T \);
(ii) if \( A \in \mathcal{F}_T \), then \( A^c \in \mathcal{F}_T \);
(iii) if \( A_1, A_2, \ldots \) is a countable family of sets in \( \mathcal{F}_T \), then \( \cup_{k \geq 1} A_k \in \mathcal{F}_T \).

We start with (i): for any fixed \( n \), \( \Omega \cup \{ T \leq n \} \in \mathcal{F}_n \) because \( T \) is a stopping time.

For (ii), assume now that \( A \in \mathcal{F}_T \). Then for any \( n \) we have \( A \cap \{ T \leq n \} \in \mathcal{F}_n \). We also have \( \{ T \leq n \} \in \mathcal{F}_n \), so
\[
A^c \cap \{ T \leq n \} = \{ T \leq n \} - A \cap \{ T \leq n \} \in \mathcal{F}_n.
\]
Finally, if for any \( i \) we have \( A_i \cap \{ T \leq n \} \in \mathcal{F}_n \), then their countable union is also in \( \mathcal{F}_n \), and the result follows from
\[
\cup_{i=1}^\infty (A_i \cap \{ T \leq n \}) = (\cup_{i=1}^\infty A_i) \cap \{ T \leq n \}.
\]

**Exercise 4.** Let \( S \) and \( T \) be stopping times for a filtration \( (\mathcal{F}_n)_{n \geq 1} \). Prove that \( \max(S, T) \) and \( \min(S, T) \) are stopping times.

**Solution.** We have
\[
\{ \max(S, T) \leq n \} = \{ S \leq n \} \cap \{ T \leq n \} \in \mathcal{F}_n,
\]
so \( \max(S, T) \) is a stopping time.

Moreover,
\[
\{ \min(S, T) \leq n \} = (\{ S \leq n \}^c \cap \{ T \leq n \})^c \in \mathcal{F}_n,
\]
so \( \min(S, T) \) is a stopping time.

**Exercise 5.** Let \( S \leq T \) be two stopping times and \( A \in \mathcal{F}_S \). Define \( U(\omega) = S(\omega) \) if \( \omega \in A \), \( U(\omega) = T(\omega) \) if \( \omega \notin A \). Prove that \( U \) is a stopping time.

**Solution.** From the definition, we have
\[
\{ U \leq n \} = (A \cap \{ S \leq n \}) \cup (A^c \cap \{ T \leq n \}).
\]
By definition of \( \mathcal{F}_S \), we have \( A \cap \{ S \leq n \} \in \mathcal{F}_n \). Moreover as \( S \leq T \) we have \( \mathcal{F}_S \subset \mathcal{F}_T \), so \( A \in \mathcal{F}_T \) and therefore \( A^c \in \mathcal{F}_T \), which implies \( A^c \cap \{ T \leq n \} \in \mathcal{F}_n \).

This concludes the proof that \( \{ U \leq n \} \in \mathcal{F}_n \).

**Exercise 6.** Consider the random walk \( S_n = \sum_{k=1}^n X_k \), the \( X_k \)'s being i.i.d., \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2 \), \( \mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n) \).

Prove that \( (S_n^2 - n, n \geq 0) \) is a \( (\mathcal{F}_n) \)-martingale. Let \( \tau \) be a bounded stopping time. Prove that \( \mathbb{E}(S_\tau^2) = \mathbb{E}(\tau) \).
Take now \( \tau = \inf \{ n \mid S_n \in \{-a, b\} \} \), where \( a, b \in \mathbb{N}^* \). Prove that \( \mathbb{E}(S_\tau) = 0 \) and \( \mathbb{E}(S_\tau^2) = \mathbb{E}(\tau) \). What is \( \mathbb{P}(S_\tau = -a) \)? What is \( \mathbb{E}(\tau) \)?

Let \( \tau' = \inf \{ n \mid S_n = b \} \). Prove that \( \mathbb{E}(\tau') = +\infty \).

**Solution.** Measurability and integrability of \( S_n^2 - n \) are elementary. For the martingale property, we simply need to write, for \( m < n \),

\[
\mathbb{E}(S_n^2 - n \mid \mathcal{F}_m) = \mathbb{E}(S_m^2 - m + (S_n - S_m)^2 + 2S_m(S_n - S_m) - n + m \mid \mathcal{F}_m)
\]

\[
= S_m^2 - m + \mathbb{E}((S_n - S_m)^2) + 2S_m \mathbb{E}(S_n - S_m) - n + m = S_m^2 - m.
\]

The equation \( \mathbb{E}(S_\tau^2) = \mathbb{E}(\tau) \) therefore follows directly from Doob’s stopping time theorem for bounded stopping times.

For \( \tau = \inf \{ n \mid S_n \in \{-a, b\} \} \), \( \min(\tau, n) \) is a bounded stopping time so from Doob’s stopping time theorem for bounded stopping times, we have

\[
\mathbb{E}(S_{\min(\tau, n)}) = \mathbb{E}(S_0) = 0.
\]

We can take \( n \to \infty \) in the equation above and obtain, by dominated convergence (note that \( S_{\min(\tau, n)} \) is uniformly bounded by \( \max(a, b) \)), \( \mathbb{E}(S_\tau) = 0 \). Important remark: for dominated convergence we need \( S_{\min(n, \tau)} \to S_\tau \) which is true only of \( \tau < \infty \) a.s., and needs to be proved separately (we have already proved in class and/or previous homework that \( T_a < \infty \) a.s., i.e. recurrence of the random walk, which implies \( \tau < \infty \) a.s.).

The equation \( \mathbb{E}(S_{\min(n, \tau)}^2) = \mathbb{E}(\min(n, \tau)) \) also give the desired result by taking \( n \to \infty \) and applying dominated convergence.

From \( \mathbb{P}(\tau = \infty) = 0 \) and \( \mathbb{E}(S_\tau) = 0 \), we have

\[
0 = \mathbb{E}(S_\tau) = -a\mathbb{P}(S_\tau = -a) + b\mathbb{P}(S_\tau = b) = -a\mathbb{P}(S_\tau = -a) + b(1 - \mathbb{P}(S_\tau = -a)).
\]

Solving the previous equation gives

\[
\mathbb{P}(S_\tau = -a) = \frac{b}{a + b}.
\]

We have

\[
\mathbb{E}(\tau) = \mathbb{E}(S_\tau^2) = a^2\mathbb{P}(S_\tau = -a) + b^2\mathbb{P}(S_\tau = b) = \frac{a^2 b}{a + b} + \frac{b^2 a}{a + b} = ab.
\]

The last question is easily answered by taking \( a \to \infty \) above and applying monotone convergence.