Probability, homework 13, due May 6.

Exercise 1. Let $X_i, i \geq 1$, be iid random variables, $X_i \geq 0, E(X_i) = 1$. Prove that if $Y_n = \prod_1^n X_k$, $\mathcal{F}_n = \sigma(X_k, k \leq n)$, $(Y_n)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Prove that if $\mathbb{P}(X_1 = 1) < 1$, Y_n converges to 0 almost surely.

Exercise 2. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, $(X_n)_{n\geq 0}$ a sequence of integrable random variables with $\mathbb{E}\left(X_n \mid \mathcal{F}_{n-1}\right) = 0$, and assume X_n is $\mathcal{F}_n$-measurable for every n. Let $S_n = \sum_{k=0}^n X_k$. Show that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

Exercise 3. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\mathcal{F}_T$ is a σ-algebra.

Exercise 4. Let S and T be stopping times for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\max(S, T)$ and $\min(S, T)$ are stopping times.

Exercise 5. Let $S \leq T$ be two stopping times and $A \in \mathscr{F}_S$. Define $U(\omega) = S(\omega)$ if $\omega \in A$, $U(\omega) = T(\omega)$ if $\omega \notin A$. prove that U is a stopping time.

Exercise 6. Consider the random walk $S_n = \sum_k^n X_k$, the X_k's being i.i.d., $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, $\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)$.

Prove that $(S_n^2 - n, n \geq 0)$ is a $(\mathcal{F}_n)-$martingale. Let τ be a bounded stopping time. Prove that $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$.

Take now $\tau = \inf\{n \mid S_n \in \{-a, b\}\}$, where $a, b \in \mathbb{N}^*$. Prove that $\mathbb{E}(S_\tau) = 0$ and $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$. What is $\mathbb{P}(S_\tau = -a)$? What is $\mathbb{E}(\tau)$?

Let $\tau' = \inf\{n \mid S_n = b\}$. Prove that $\mathbb{E}(\tau') = +\infty$.