Exercise 1. Kolmogorov’s maximal inequality and convergence of random series. Let \((X_n)_{n \geq 1}\) be a sequence of mutually independent random variables, on the same probability space, with expectation 0 and finite variance. Let \(S_n = \sum_{\ell=1}^{n} X_\ell\). Prove that for any \(\lambda > 0\),
\[
\lambda^2 \mathbb{P}(\max_{1 \leq k \leq n} |S_k| \geq \lambda) \leq \text{Var}(S_n).
\]
Prove that if \(\sum_\ell \text{Var}(X_\ell) < \infty\), then \((S_n)_{n \geq 1}\) converges almost surely.

Exercise 2. Let \(Y\) be an integrable random variable on \((\Omega, \mathcal{A}, \mathbb{P})\) and \(G\) a sub-\(\sigma\)-algebra of \(\mathcal{A}\). Show that
\[
|\mathbb{E}(Y \mid G)| \leq \mathbb{E}(|Y| \mid G).
\]

Exercise 3. Let \(Y\) be an integrable random variable on \((\Omega, \mathcal{A}, \mathbb{P})\) and \(G\) a sub-\(\sigma\)-algebra of \(\mathcal{A}\). Suppose that \(H \subset G\) is a sub-\(\sigma\)-algebra of \(G\). Show that \(\mathbb{E}(\mathbb{E}(Y \mid G) \mid H) = \mathbb{E}(Y \mid H)\).

Exercise 4. Let \((X_n)_{n \geq 1}\) be independent such that \(\mathbb{E}(X_i) = m_i\), \(\text{var}(X_i) = \sigma_i^2\), \(i \geq 1\). Let \(S_n = \sum_{i=1}^{n} X_i\) and \(\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)\).
   a) Find sequences \((b_n)_{n \geq 1}\), \((c_n)_{n \geq 1}\) of real numbers such that \((S_n^2 + b_n S_n + c_n)_{n \geq 1}\) is a \((\mathcal{F}_n)_{n \geq 1}\)-martingale.
   b) Assume moreover that there is a real number \(\lambda\) such that \(e^{\lambda X_1} \in \mathbb{L}^1\) for any \(i \geq 1\). Find a sequence \((a_n^{(\lambda)})_{n \geq 1}\) such that \((e^{\lambda S_n - a_n^{(\lambda)}})_{n \geq 1}\) is a \((\mathcal{F}_n)_{n \geq 1}\)-martingale.

Exercise 5. Let \((X_k)_{k \geq 0}\) be i.i.d. random variables, \(\mathcal{F}_m = \sigma(X_1, \ldots, X_m)\) and \(Y_m = \prod_{k=1}^{m} X_k\). Under which conditions is \((Y_m)_{m \geq 1}\) a \((\mathcal{F}_m)_{m \geq 1}\)-submartingale, supermartingale, martingale?

Exercise 6. Let \(a > 0\) be fixed, \((X_i)_{i \geq 1}\) be iid, \(\mathbb{R}^d\)-valued random variables, uniformly distributed on the ball \(B(0, a)\). Set \(S_n = x + \sum_{i=1}^{n} X_i\).
   a) Let \(f\) be a superharmonic function. Show that \((f(S_n))_{n \geq 1}\) defines a supermartingale.
   b) Prove that if \(d \leq 2\) any nonnegative superharmonic function is constant. Does this result remain true when \(d \geq 2\)?