Exercise 1. Sufficient condition for convergence in distribution. Assume that the sequence of random variables \((X_n)_{n \geq 1}\), \(X\) are such that
\[
\mathbb{E} f(X_n) \to \mathbb{E} f(X)
\]
or any smooth and compactly supported function \(f\). Prove that \(X_n\) converges to \(X\) in distribution.

Solution. Let \(u\) be fixed and \(g(x) = e^{iu x}\). Let \(\chi\) be a smooth cutoff function, i.e. \(\chi(x) = 0\) for \(|x| \in [2, \infty)\), \(\chi(x) = 1\) for \(|x| \in [0, 1]\). Then for any \(M > 0\)
\[
\mathbb{E} g(X_n) = \mathbb{E}(g(X_n)\chi(X_n/M)) + \mathbb{E}(g(X_n)(1 - \chi(X_n/M))) \tag{0.1}
\]
Let \(\varepsilon > 0\) be fixed. By monotone convergence there exists \(M_0\) such that
\[
\mathbb{E}\chi(X_n/M_0) > 1 - \varepsilon. \tag{0.2}
\]
From the exercise hypothesis, we have \(\mathbb{E}(\chi(X_n/M_0)) \to \mathbb{E}(\chi(X/M_0))\) so there exists \(n_0\) such that for any \(n > n_0\) we have \(\mathbb{E}(\chi(X_n/M_0))) > 1 - 2\varepsilon\), so \(\mathbb{E}(1 - \chi(X_n/M_0)) < 2\varepsilon\).

The above paragraph and (0.1) give, for any \(n \geq n_0\),
\[
|\mathbb{E}(\chi(X_n/M_0))| > 1 - \varepsilon.
\]
Moreover, \(x \mapsto g(x)\chi(x/M_0)\) is smooth and compactly supported so \(\mathbb{E}(g(X_n)\chi(X_n/M_0)) \to \mathbb{E}(g(X)\chi(X/M_0))\) as \(n \to \infty\). So there is a \(n_1 > n_0\) such that for any \(n > n_1\) we have
\[
|\mathbb{E} g(X_n) - \mathbb{E}(g(X)\chi(X/M_0))| \leq 3\varepsilon.
\]
Together with (0.2) this implies that for any \(n > n_1\) we have
\[
|\mathbb{E} g(X_n) - \mathbb{E} g(X)| \leq 4\varepsilon.
\]
As \(\varepsilon\) is arbitrary, we have proved \(\mathbb{E} g(X_n) \to \mathbb{E} g(X)\), i.e. convergence of characteristic functions, hence convergence in distribution.

Exercise 2. Assume that the sequence of random variables \((X_n)_{n \geq 1}\) satisfies \(\mathbb{E} X_n \to 1\) and \(\mathbb{E} X_n^2 \to 1\). Prove that \((X_n)_{n \geq 1}\) converges in distribution. What is the limit?

Solution. We have
\[
\mathbb{E}((X_n - 1)^2) = \mathbb{E}(X_n^2) - 2\mathbb{E}(X_n) + 1 \to 1 - 2 + 1 = 0,
\]
so \(X_n \to 1\) in \(L^2\), hence also in distribution.

Exercise 3. Convergence in \(L^1\) in the strong law of large numbers.

a) Read online (or in Jacod-Protter) the definition of a uniformly integrable sequence of random variables.

b) Prove that if \(S_n\) converges to \(S\) almost surely, and \((S_n)_{n \geq 1}\) is uniformly integrable, then \(S_n\) converges to \(S\) in \(L^1\).
c) Prove that if the $X_i$’s are i.i.d. and in $L^1$, then $(n^{-1} \sum_{k=1}^{n} X_k)_{n \geq 1}$ is uniformly integrable.

d) Conclude that the strong law of large numbers in the almost sure sense for random variables in $L^1$ implies the strong law of large numbers in the $L^1$ sense.

**Solution.**

b) As $(S_n)_{n \geq 1}$ is uniformly integrable, we have

$$\lim_{K \to \infty} \sup_n \mathbb{E}(|S_n| I_{|S_n| > K}) = 0.$$  \hspace{1cm} (0.3)

We can write

$$\mathbb{E}|S_n - S| \leq \mathbb{E}|S_n| I_{|S_n| > K} + \mathbb{E}|S| I_{|S| > K} + \mathbb{E}|S_n - S| I_{|S_n| < K, |S| < K}.$$  

Let $\varepsilon > 0$. By uniform integrability, there exists $K_1$ such that the first term on the RHS is smaller than $\varepsilon$ for any $n$. Moreover, by Fatou $S$ is in $L^1$, so by monotone convergence there exists $K_2 > K_1$ such that the second term is also smaller than $\varepsilon$. For such a $K_2$, the third term converges to 0 (and therefore gets smaller than $\varepsilon$) as $n \to \infty$. Hence we have proved that there exists $n_0$ such that for any $n > n_0$ we have

$$\mathbb{E}|S_n - S| \leq 3\varepsilon.$$  

This concludes the proof of convergence in $L^1$.

c) This is a difficult question. You first can prove that uniform integrability of any sequence $(X_n)_n$ is equivalent to both of the following conditions being satisfied:

- $\sup_n \mathbb{E}|X_n| < \infty$
- for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A$ with $\mathbb{P}(A) < \delta$ we have $\sup_n \mathbb{E}|X_n| I_A < \varepsilon$

Then it is clear that uniform integrability for a sequence of random variables implies uniform integrability for their empirical means.

d) This is immediate from the previous questions.

**Exercise 4.** Let $(X_n)_{n \geq 1}$ be a sequence of random variables, on the same probability space, with $\mathbb{E}(X_i) = \mu$ for any $\ell$, and a weak correlation in the following sense: $\text{Cov}(X_k, X_\ell) \leq f(|k - \ell|)$ for all indexes $k, \ell$, where the sequence $(f(m))_{m \geq 0}$ converges to 0 as $m \to \infty$. Prove that $(n^{-1} \sum_{k=1}^{n} X_k)_{n \geq 1}$ converges to $\mu$ in $L^2$.

**Solution.** Without loss of generality we can assume $\mu = 0$. Then expansion gives

$$\mathbb{E}\left((n^{-1} \sum_{i=1}^{n} X_i)^2\right) \leq n^{-2} \sum_{1 \leq k, \ell \leq n} f(|k - \ell|) \leq n^{-1} \sum_{0 \leq i \leq n} |f(i)|$$

As $f$ converges to 0, the above RHS also converges to 0, which concludes the proof.

**Exercise 5** Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables, on the same probability space, with law given by $\mathbb{P}(X_1 = (-1)^m m) = 1/(cm^2 \log m)$ for $m \geq 2$ ($c$ is the normalization constant $c = \sum_{m \geq 2} 1/(m^2 \log m)$). Prove that $\mathbb{E}(|X_1|) = \infty$, but there exists a constant $\mu \notin \{\pm \infty\}$ such that $(n^{-1} \sum_{k=1}^{n} X_k)_{n \geq 1}$ converges to $\mu$ in probability. Does it converge almost surely, and in $L^p$?
Solution. First question. We have
\[
E(|X_1|) = c^{-1} \sum_{m \geq 2} \frac{1}{m \log m} \geq c^{-1} \int_2^\infty \frac{dx}{x \log x} = +\infty
\]

Second question. Let \(\varepsilon > 0\) and \(Y_k = X_k \mathbb{1}_{|X_k| \leq n}\). First,
\[
E(Y_1) = \sum_{m=2}^n \frac{(-1)^m}{m \log m} \to \mu \tag{0.4}
\]
where \(\mu\) is a finite number (this convergence follows from partial sums of alternating series).

Then by the union bound
\[
P(\exists k \in [1,n] : X_k \neq Y_k) \leq nP(|X_1| > n) \leq nC_1 \int_n^\infty \frac{dx}{x^2 \log x} \leq nC_2 \frac{1}{n \log n} \to 0, \tag{0.5}
\]
where \(C_1, C_2\) are some universal constant.

Finally,
\[
P(\sqrt{n^{-1} \sum_{k=1}^n (Y_k - E(Y_k))} > \varepsilon) \leq \varepsilon^{-2} n^{-2} E((\sum_{k=1}^n (Y_k - E(Y_k)))^2) = n^{-1} E((Y_1 - E(Y_1))^2)
\leq E(Y_1^2)/n + 2\mu/n \leq n^{-1} \sum_{m=2}^n \frac{1}{m \log m} + \frac{2\mu}{n} \to 0, \tag{0.6}
\]
where the first inequality holds for large enough \(n\), due to (0.4), and follows from \((a - b)^2 \leq 2(a^2 + b^2)\).

From (0.4), (0.5) and (0.6) one concludes easily that \(n^{-1} \sum_{k=1}^n X_k \to \mu\) in probability.

Third question. One can check easily that for some universal constant \(c_1\), we have \(\sum_{k=1}^n \mathbb{P}(|X_k| > k) \geq c_1 \sum_{k=1}^n (k \log k)^{-1} = +\infty\). From the Borel Cantelli lemma for independent events, this implies
\[
\mathbb{P}(|X_k| > k \text{ i.o.}) = 1.
\]
On the other hand, if \(|X_k| > k \text{ i.o.}\), for example along a sequence \(k_1, k_2, \ldots\) of indices,
\[
\left| \sum_{j=1}^{k_\ell-1} X_j - \sum_{j=1}^{k_\ell} X_j \right| > k_\ell
\]
and division by \(k_\ell\) and \(\ell \to \infty\) in the above inequality contradicts the convergence \(n^{-1} \sum_{j=1}^n X_j \to \mu\). So there is no almost sure convergence.

Fourth question. There is obviously no convergence in \(L^1\) (and therefore \(L^q\) for \(q \geq 1\)) because the random variables are not in \(L^1\).

From (0.4) and (0.5) we know that \(n^{-1} \sum_{k=1}^n Y_k\) converges to \(\mu\) in \(L^2\), hence in \(L^p\) also for any \(0 < p < 2\). Hence to prove convergence of \(n^{-1} \sum_{k=1}^n X_k\) in \(L^p\) for any \(0 < p < 1\) we just need to prove
\[
E(n^{-1} \sum_{k=1}^n (X_k - Y_k)^p) \to 0
\]
For any positive $a_i$’s and $(p \in [0,1])$ we have $\left(\sum a_i\right)^p \leq \sum a_i^p$ so we just need to prove
\[ \mathbb{E}|X_1 - Y_1|^p \to 0 \]
This is an easy calculation as the above LHS is of order
\[ \sum_{k=n+1}^{\infty} \frac{k^p}{k^2 \log k} \to 0 \]
where we critically use $0 < p < 1$ for the last inequality.

**Exercise 6.** Let the $X_i$’s be independent uniformly bounded real random variables. Let $\mu_\ell = \mathbb{E}(X_\ell)$, and $\sigma^2_\ell = \text{Var}(X_\ell)$ satisfy $c_1 < \sigma^2_\ell$ for some $c_1$ which does not depend on $\ell$. State and prove a central limit theorem for $\sum_{\ell=1}^{n} X_\ell$.

**Solution.** This exercise is really about adapting the proof of the CLT either by Lindeberg’s method or characteristic functions. All steps are the same as in class, with key observation that as the $X_i$’s are uniformly bounded, so is their 3rd moment so that all estimates are easy. The result is
\[ \left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)/\left(\sum_{i=1}^{n} \sigma^2_i\right)^{1/2} \to \mathcal{N}(0, 1) \]
in distribution.