Probability, homework 1, due February 12.

Exercise 1.

(i) A family has 5 children, consisting of 3 girls and 2 boys. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are girls?

(ii) How many ways are there to split 11 people into 3 teams, where one team has 2 people, one has 4 and the other 5?

Exercise 2. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}.$$

Exercise 3. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \ n \geq k.$$

Give a combinatorial argument (no computations are needed) to establish this identity. *Hint*: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

Exercise 4. Proof of Stirling's formula. Using a trapezoidal approximation for the area under a curve, prove that

$$\sum_{k=1}^{n} \log k = \left(n + \frac{1}{2}\right) \log n - n + c + \mathrm{o}(1)$$

as $n \to \infty$, where c is a non-explicit constant.