Probability, homework 9, due November 22nd.

Some exercises are from *A first course in probability*, ninth edition, by Sheldon Ross.

Exercise 1. The joint density of X and Y is given by $f(x,y) = \frac{e^{-y}}{y}$, $0 < x < y$, $0 < y < \infty$. Compute $\mathbb{E}(X^3 \mid Y = y)$.

Exercise 2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed positive random variables. For $k \leq n$, find

$$\mathbb{E}\left(\frac{\sum_{i=1}^k X_i}{\sum_{i=1}^n X_i}\right).$$

Exercise 3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that $X_1 \geq X_2 \geq \cdots \geq X_{N-1}$ and $X_{N-1} < X_N$. That is, N is the point at which the sequence stops decreasing. Show that $\mathbb{E}(N) = e$.

Hint: First find $\mathbb{P}(N \geq n)$.

Exercise 4. Let X be a random variable with density $f_X(x) = (1 - |x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1 - \cos u)}{u^2}.$$

Exercise 5. Let X have density $\frac{1}{2}e^{-|x|}$. What is the characteristic function of X?

Exercise 6. Let X_λ be a real random variable, with Poisson distribution with parameter λ. Calculate the characteristic function of X_λ. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Exercise 7. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$