Probability, homework 8, due November 15th.

Some exercises are from *A first course in probability*, by Sheldon Ross.

Exercise 1. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2;

(b) X_1, X_2, X_3.

Exercise 2. The number of people who enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

Exercise 3. Let X be a random variable uniform on $[0, 1]$.

(i) Calculate $\mathrm{Var}(X)$.

(ii) What is the density of $1/X$?

Exercise 4. The random variables X and Y have joint density function $f(x, y) = 12xy(1 - x)$, $0 < x < 1$, $0 < y < 1$, and equal to 0 otherwise.

(a) Are X and Y independent?

(b) Find $\mathbb{E}[X]$.

(c) Find $\mathbb{E}[Y]$.

(d) Find $\mathrm{Var}(X)$.

(e) Find $\mathrm{Var}(Y)$.

Exercise 5. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Define $Z_k = \prod_{i=1}^{k} X_i$. Prove that $(Z_k)_{k\geq 1}$ is a sequence of independent random variables.

Exercise 6. Let (X, Y) be uniform on the unit ball, i.e. it has density

$$f_{(X,Y)}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if} \quad x^2 + y^2 \leq 1, \\ 0 & \text{if} \quad x^2 + y^2 > 1. \end{cases}$$

Find the density of $\sqrt{X^2 + Y^2}$.

Exercise 7. Let X, Y be random variables.

(a) If the distribution of (X, Y) is $\lambda\mu e^{-\lambda x-\mu y}\mathbb{1}_{\mathbb{R}_+^2}(x, y)$, what is the distribution of $\min(X, Y)$?

(b) If the distribution of (X, Y) is $\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$, what is the distribution of X/Y?

Exercise 8. Let $X_1 \leq X_2 \leq \cdots \leq X_n$ be the ordered values of n independent uniform random variables on $[0, 1]$. Prove that for $1 \leq k \leq n + 1$,

$$\mathbb{P}\left(X_k - X_{k-1} > t\right) = (1 - t)^n$$

where $X_0 = 0$, $X_{n+1} = 1$.