Probability, homework 3, due September 27th.

From *A first course in probability*, ninth edition, by Sheldon Ross.

Exercise 1. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

Exercise 2. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be
(a) no complete pair?
(b) exactly 1 complete pair?

Exercise 3. Show that the probability that exactly one of the events E or F occurs equals $\mathbb{P}(E) + \mathbb{P}(F) - 2\mathbb{P}(E \cap F)$.

Exercise 4. If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? Compute for all values of i between 2 and 12.

Exercise 5. Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random. What is the probability that the marble is black? What is the probability that the first box was the one selected given that the marble is white?

Exercise 6. Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately 0.135, increasing to approximately 0.268 if the man does have cancer. If, on the basis of other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that
(a) the test indicated an elevated PSA level?
(b) the test did not indicate an elevated PSA level?

Exercise 7. Show that if $\mathbb{P}(A) > 0$, then $\mathbb{P}(A \cap B \mid A) \geq \mathbb{P}(A \cap B \mid A \cup B)$.

Exercise 8. Let Q_n denote the probability that no run of 3 consecutive heads appears in n tosses of a fair coin. Show that

$$Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3},$$

and $Q_0 = Q_1 = Q_2 = 1$. Find Q_8.