Probability, final exam practice.

The final will consist in eight such exercises.

Exercise 1. Evaluate the following explicitely:

$$\sum_{k=1}^{364} \frac{365!}{k!(365-k)!}\frac{1}{7^k},$$

$$\sum_{a+b+c+d=12} \frac{13!}{a!b!c!d!},$$

where the above sum is over all nonnegative integers a, b, c, d (such that $a+b+c+d = 12$).

Exercise 2. The forecast predicts either a sunny day (event S) or a rainy day (event S^c), and I will either wear flip flops (event F) or not (event F^c). Assume that 70% of the days are sunny and that I am "80% accurate" in the sense that $\mathbb{P}(F \mid S) = \mathbb{P}(F^c \mid S^c) = 8/10$.

Find the probability that it is a sunny day, given that I wear flip flops.

Exercise 3. It is raining. I can see exactly a billion independent droplets fall on the ground. Each droplet has probability $1/10^8$ to fall on my head. Let X be the number of droplets falling on my head. What is exactly $\mathbb{E}(X)$? What is approximately $\mathrm{Var}(X)$. What is approximately $\mathbb{P}(X > 100)$?

Exercise 4. Nine people enter a coffee shop. Each of them will order a cappuccino, independently of the others, with probability $1/4$. Let X be the number of ordered cappuccinos. Compute the following: $\mathbb{E}(X)$, and the variance of X.

Exercise 5. Let X be a continuous random variable with density cx^2 on $[0, 1]$, c on $[1, 2]$, and 0 otherwise. Find c. Calculate the expectation of X and its cummulative distribution function.

Exercise 6. Let X and Y have joint density $\frac{12}{7}x(x+y)$ on $0 \leq x, y \leq 1$. Calculate the density of X. Calculate the density of X knowing $Y = y$. What is the covariance between X and Y?

Exercise 7. Let X be a Gaussian random variable with mean 1 and variance 2, and Y an independent Gaussian random variable with mean 2 and variance 3. What type of random variable is $X + Y$? What is its density?

Exercise 8. Let X be an exponential random variable with parameter 1 and Y an independent exponential random variable with parameter 2. What type of random variable is $\min(X, Y)$? With which parameter? Prove it.

Exercise 9. Let U be a uniform random variable on $[0, 1]$. What type of random variable is $1/U$? Prove it.

Exercise 11. Let $U_1, \ldots, U_n, \ldots$ be iid uniform random variable on $[0,1]$. State the weak law of large numbers for their sum. What is the limiting probability that $\sum_1^n U_1 > 2n/3$ as $n \to \infty$?

Exercise 11. Let $U_1, \ldots, U_n, \ldots$ be iid uniform random variable on $[0,1]$. State the central limit theorem for their sum. What is the limiting probability that $\sum_1^n U_1 - n/2 > 4\sqrt{n}$ as $n \to \infty$?