Probability, homework 9, due December 9.

Exercise 1. Let X_λ be a real random variable, with Poisson distribution with parameter λ. Calculate the characteristic function of X_λ. Conclude that $(X_\lambda-\lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda\to\infty$.

Exercise 2. Find an example of real random variables $(X_n)_{n\geq 1}$, X, in L^1, such that $(X_n)_{n\geq 1}$ converges to X in distribution and $\mathbb{E}(X_n)$ converges, but not towards $\mathbb{E}(X)$.

Exercise 3. Assume a probability space $(\Omega,\mathcal{A},\mathbb{P})$ is such that Ω is countable and $\mathcal{A}=2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. Bernoulli random variables, on the same probability space, with parameter $1/2$ ($\mathbb{P}(X_n=0)=\mathbb{P}(X_n=1)=1/2$), and let τ_n be the hitting time of level n by the partial sums, i.e. $\tau_n=\inf\{k\mid \sum_{\ell=1}^k X_\ell=n\}$. Show that $n^{-1}\tau_n$ converges to 2 almost surely.

Exercise 5. The problem of the collector. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on $\{1,\dots,n\}$. Let $\tau_n=\inf\{m\geq 1:\{X_1,\dots,X_m\}=\{1,\dots,n\}\}$ be the first time for which all values have been observed.

(i) Let $\tau_n^{(k)}=\inf\{m\geq 1:|\{X_1,\dots,X_m\}|=k\}$. Prove that the random variables $(\tau_n^{(k)}-\tau_n^{(k-1)})_{2\leq k\leq n}$ are independent and calculate their respective distributions.

(ii) Deduce that $\frac{\tau_n}{n\log n}\to 1$ in probability as $n\to\infty$, i.e. for any $\varepsilon>0$,

$$\mathbb{P}\left(\left|\frac{\tau_n}{n\log n}-1\right|>\varepsilon\right)\to 0.$$

Exercise 6. The number of buses stopping till time t. Let $(X_n)_{n\geq 1}$ be i.i.d random variables on $(\Omega,\mathcal{A},\mathbb{P})$, X_1 being an exponential random variable with parameter 1. Define $T_0=0$, $T_n=X_1+\dots+X_n$, and for any $t>0$, $N_t=\max\{n\geq 0\mid T_n\leq t\}$.

(i) For any $n\geq 1$, calculate the joint distribution of $(T_1,\dots,T_n)$.

(ii) Deduce the distribution of N_t, for arbitrary t.