Probability, homework 8, due December 3.

Exercise 1. Let X be a random variable with density $f_X(x) = (1-|x|)\mathbb{1}_{(-1,1)}(x)$. Show that its characteristic function is

$$\phi_X(u) = \frac{2(1-\cos u)}{u^2}.$$

Exercise 2. Let f be a continuous fuction on $\mathbb{R}$, and assume that $(X_n)_{n\geq 1}$ converges to X in distribution. Prove that $(f(X_n))_{n\geq 0}$ converges to $f(X)$ in distribution.

Exercise 3. Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be real random variables, with X_n and Y_n independent for any $n \geq 1$, and assume that X_n converges in distribution to X and Y_n to Y. Prove that $X_n + Y_n$ converges in distribution to $X + Y$.

Exercise 4. Prove that if a sequence of real random variables (X_n) converge in distribution to X, and (Y_n) converges in distribution to a constant c, then $X_n + Y_n$ converges in distribution to $X + c$.

Exercise 5. Let the X_ℓ's be i.i.d. with mean 0 and variance $0 < \sigma^2 < \infty$. Does $n^{-\alpha}\sum_{\ell=1}^n X_\ell$ converge in distribution for $0 < \alpha < 1/2$? Same question for $\alpha = 1/2$ and $\alpha > 1/2$.

Exercise 6. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with standard Cauchy distribution and let $M_n = \max(X_1, \dots, X_n)$. Prove that $(nM_n^{-1})_{n\geq 1}$ converges in distribution and identify the limit.

Exercise 7. Let the X_ℓ's be independent uniformly bounded real random variables. Let $\mu_\ell = \mathbb{E}(X_\ell)$, and $\sigma_\ell^2 = \mathrm{Var}(X_\ell)$ satisfy $c_1 < \sigma_\ell^2$ for some c_1 which does not depend on ℓ. State and prove a central limit theorem for $\sum_{\ell=1}^n X_\ell$.

Exercise 8. Let $(X_n)_{n\geq 1}$ be a sequence of independent real random variables, all uniformly distributed on $[0, 1]$. Does $n \inf(X_1, \dots, X_n)$ converge in law as $n \to \infty$? If yes, what is the limiting distribution?

Exercise 9. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0, 1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

(i) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^n X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

(ii) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$ and conclude.

Exercise 10. Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. For $a, b \in \mathbb{Z}$, let $T_a = \inf\{n \geq 0 : s_n = a\}$ and $T_{a,b} = \inf\{n \geq 0 : s_n = a \textit{ or } s_n = b\}$. For $x \in \mathbb{Z}$, let $\omega(x) = \mathbb{P}(s_{T_{a,b}} = b \mid s_0 = x)$.

Prove that for $a < x < b$, $\omega(x) = \frac{1}{2}(\omega(x+1) + \omega(x-1))$, provided we define $\omega(a) = 0$ and $\omega(b) = 1$. Conclude that

$$\omega(x) = \frac{x - a}{b - a}.$$

From this result, prove that $\mathbb{P}(T_b < \infty) = 1$.