Probability, homework 7, due November 11.

Exercise 1. Let X be uniform on $(-\pi,\pi)$ and $Y=\sin(X)$. Show that the density of Y is
$$\frac{1}{\pi\sqrt{1-y^2}}\mathbb{1}_{[-1,1]}(y).$$

Exercise 2. Let (X,Y) be uniform on the unit ball, i.e. it has density
$$f_{(X,Y)}(x,y)=\begin{cases}\frac{1}{\pi} & \text{if} \quad x^2+y^2\leq 1,\\ 0 & \text{if} \quad x^2+y^2>1.\end{cases}$$
Find the density of $\sqrt{X^2+Y^2}$.

Exercise 3. In the (O,x,y) plane, a random ray emerges from a light source at point $(-1,0)$, towards the (O,y) axis. The angle with the (O,x) axis is uniform on $(-\frac{\pi}{2},\frac{\pi}{2})$. What is the distribution of the impact point with the (O,y) axis?

Exercise 4. On a probability space $(\Omega,\mathcal{A},\mathbb{P})$ is given a random variable (X,Y) with values in $\mathbb{R}^2$.

(a) If the law of (X,Y) is $\lambda\mu e^{-\lambda x-\mu y}\mathbb{1}_{\mathbb{R}_+^2}(x,y)\mathrm{d}x\mathrm{d}y$, what is the law of $\min(X,Y)$?

(b) If the law of (X,Y) is $\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}\mathrm{d}x\mathrm{d}y$, what is the law of X/Y?

Exercise 5. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that
$$\lim_{n\to\infty}\frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2}=\frac{1}{4}\ \text{a.s.}$$

Exercise 6. Assume that $X_1,X_2,\ldots$ are independent random variables uniformly distributed on $[0,1]$. Let $Y^{(n)}=n\inf\{X_i, 1\leq i\leq n\}$. Prove that it converges weakly to an exponential random variable, i.e. for any continuous bounded function $f:\mathbb{R}^+\to\mathbb{R}$,
$$\mathbb{E}\left(f(Y^{(n)})\right)\underset{n\to\infty}{\longrightarrow}\int_{\mathbb{R}^+}f(u)e^{-u}\mathrm{d}u.$$