Probability, homework 5, due October 14.

Exercise 1. Calculate $\mathbb{E}(X)$ for the following probability measures $\mathbb{P}^X$.

(i) $\mathbb{P}^X = p\delta_a + q\delta_b$ where $p + q = 1$, $p, q \geq 0$ and $a, b \in \mathbb{R}$;

(ii) $\mathbb{P}^X$ is the Poisson distribution: $\mathbb{P}^X(\{n\}) = e^{-\lambda}\frac{\lambda^n}{n!}$ for any integer $n \geq 0$, for some $\lambda > 0$.

Exercise 1. Let X be uniformly distributed on $[0, 1]$ and $\lambda > 0$. Show that $-\lambda^{-1} \log X$ has the same distribution as an exponential random variable with parameter λ.

Exercise 3. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 4. Suppose that a fair coin is tossed N times, with all outcomes (i.e. sequences of N elements in $\{H, T\}$) being equiprobable. Let $X_i \in \{H, T\}$ be the outcome of the ith coin toss.

(i) What are $\Omega, \mathcal{A}$ and $\mathbb{P}$?

(ii) Let p_N be the probability that the pattern (H, H, T, H, T, H, H) occurs at some point in the sequence $(X_i)_{i=1}^N$. What is the limit of p_N as $N \to \infty$?

Exercise 5. Let X be a real random variable in $\mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n\geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_N) \underset{n\to\infty}{\longrightarrow} 0$. Prove that $\mathbb{E}(X\mathbb{1}_{A_n}) \underset{n\to\infty}{\longrightarrow} 0$.

Exercise 6. Let $c > 0$ and X be a real random variable such that for any $\lambda \in \mathbb{R}$

$$\mathbb{E}\left(e^{\lambda X}\right) \leq e^{c\frac{\lambda^2}{4}}.$$

Prove that, for any $\delta > 0$,

$$\mathbb{P}\left(|X| > \delta\right) \leq 2e^{-\frac{\delta^2}{c}}.$$