Probability, homework 3, due September 30.

Exercise 1. Let $\mathcal{A}$ be a σ-algebra, $\mathbb{P}$ a probability measure and $(A_n)_{n\geq 1}$ (resp. $(B_n)_{n\geq 1}$) be a sequence of events in $\mathcal{A}$ which converges to A (resp. B). Assume that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0$ for all n. Show that

(i) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B) = \mathbb{P}(A \mid B)$;
(ii) $\lim_{n\to\infty} \mathbb{P}(A \mid B_n) = \mathbb{P}(A \mid B)$;
(iii) $\lim_{n\to\infty} \mathbb{P}(A_n \mid B_n) = \mathbb{P}(A \mid B)$.

Exercise 2. Let A, B, C be three mutually independent events and $\mathbb{P}(B \cap C) \neq 0$. Prove that $\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A)$.

Exercise 3. The probability that a male driver makes an insurance claim in any given year is 0.3, while the probability that a female driver makes an insurance claim in any given year is 0.2. Furthermore, claims by the same driver in successive years are independent events. We assume equal numbers of male and female drivers.

What is the probability that a randomly chosen driver makes a claim in the first year (event A)? What is the probability that a randomly chosen driver makes a claim in the first and second years (event B)?

What is $\mathbb{P}(B \mid A)$, the probability that a randomly chosen driver makes a claim in the second year, conditionally to the fact that he/she made one on the first year? How can you explain that it is different from $\mathbb{P}(A)$ although claims in successive years are independent? If you are the head of an insurance company and want one more client, would you prefer one who had a claim the previous year or the contrary?

Exercise 4. A simple given property is genetically encoded as a pair of alleles a and A, which yields three possible genotypes $\{A, A\})$, $\{a, a\}$, and $\{A, a\}$, represented in the population with respective probabilities p, q, r, $p + q + r = 1$, homogenously and with these same probabilities for each gender. A parent passes on one of its alleles, chosen at random uniformly, to its child; the genotype of the child combines alleles from both parents.

We assume that these coefficients p, q, r are stable from one generation to another. Explain why they actually depend on only one parameter:

$$\begin{cases} p = P^2, \\ q = Q^2, \\ r = 2PQ, \end{cases}$$

where $P + Q = 1$.

Exercise 5. Let X be a random variable with Poisson distribution with paameter $\lambda > 0$. What is $\mathbb{E}(X)$? What is $\mathrm{Var}(X)$? Can you calculate the moment $\mathbb{E}(X^k)$ for any $k \in \mathbb{N}$? Hint: first calculate $\mathbb{E}(X(X-1)\dots(X-k+1))$ for any $k \in \mathbb{N}$.

Exercise 6. Let X be a geometric random variable (i.e. X has vales in $\mathbb{N}$ and $\mathbb{P}(X = i) = (1-q)^i q$ for some fixed $q \in (0,1)$). Prove the following memoryless property: for $i, j > 0$,

$$\mathbb{P}(X > i + j \mid X \geq i) = \mathbb{P}(X > j).$$