Probability, homework 2, due September 23.

Exercise 1. Suppose that Ω is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have finite complement. Prove that $\mathcal{A}$ is not a σ-algebra.

Exercise 2. Let $(A_n)_{n\geq 0}$ be a set of pairwise disjoint events and $\mathbb{P}$ a probability. Show that $\lim_{n\to\infty}\mathbb{P}(A_n)=0$.

Exercise 3. Prove the Bonferroni inequalities: if $A_i\in\mathcal{A}$ is a sequence of events, then

$$\mathbb{P}\left(\cup_{i=1}^n A_i\right)\geq\sum_{i=1}^n\mathbb{P}(A_i)-\sum_{i<j}\mathbb{P}(A_i\cap A_j).$$

Exercise 4. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: consider the event E_n that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $(n-1)$ rolls.

Exercise 5. Let $\mathbb{P}$ be a probability measure on Ω endowed with a σ-algebra $\mathscr{A}$.

(i) What is the meaning of the following events, where all A_n's are elements of $\mathscr{A}$?

$$\liminf_{n\to\infty}A_n=\bigcup_{n\geq1}\bigcap_{k\geq n}A_k,\quad \limsup_{n\to\infty}A_n=\bigcap_{n\geq1}\bigcup_{k\geq n}A_k.$$

(ii) In the special case $\Omega=\mathbb{R}$ and $\mathcal{A}$ is its Borel σ-algebra, for any $p\geq1$, let

$$A_{2p}=\left[-1,2+\frac{1}{2p}\right),\quad A_{2p+1}=\left(-2-\frac{1}{2p+1},1\right].$$

What are $\liminf_{n\to\infty}A_n$ and $\limsup_{n\to\infty}A_n$?

(iii) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n\to\infty}A_n\right)\leq\liminf_{n\to\infty}\mathbb{P}\left(A_n\right),\mathbb{P}\left(\limsup_{n\to\infty}A_n\right)\geq\limsup_{n\to\infty}\mathbb{P}\left(A_n\right).$$

Exercise 6. Let n and m be random numbers chosen independently and uniformly on $[\![1,N]\!]$. What are $\Omega,\mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on N) ? Prove that $\mathbb{P}(n\wedge m=1)\underset{N\to\infty}{\longrightarrow}\zeta(2)^{-1}$ where $\zeta(2)=\prod_{p\in\mathcal{P}}(1-p^{-2})^{-1}=\sum_{n\geq1}n^{-2}=\frac{\pi^2}{6}$ (you don't have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n\wedge m=1$ means that their greatest common divisor is 1.