Probability, midterm.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $A, B \in \mathcal{A}$. Prove that if $A \cap B = \emptyset$ and A, B are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Exercise 2. The probability that a male driver makes an insurance claim in any given year is α, while the probability that a female driver makes an insurance claim in any given year is β, and $\alpha > \beta$. Furthermore, claims by the same driver in successive years are independent events. We assume equal numbers of male and female drivers.

What is the probability that a randomly chosen driver makes a claim in the first year (event A)? What is the probability that a randomly chosen driver makes a claim in the first and second years (event B)?

What is $\mathbb{P}(B \mid A)$, the probability that a randomly chosen driver makes a claim in the second year, conditionally to the fact that he/she made one on the first year? How can you explain that it is different from $\mathbb{P}(A)$ although claims in successive years are independent?

Exercise 3. Let X be a positive random variable with density $e^{-x}\mathbb{1}_{x>0}$ (the exponential distribution). What is the density of $1/(1+X)$?

Exercise 4 Let X and Y be independent random variables uniform on $[0,1]$. What is $\mathbb{E}(|X-Y|)$?

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables. Assume that $\sum_{n=1}^{\infty} \mathbb{E}(|X_n|) < \infty$. Prove that $\sum_{n=1}^{\infty} X_n$ converges almost surely and

$$\mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(X_n\right).$$

Exercise 6 Let the X_ℓ's be i.i.d. with mean 0 and variance $0 < \sigma^2 < \infty$. Does $n^{-\alpha} \sum_{\ell=1}^{n} X_\ell$ converge in distribution for $0 < \alpha < 1/2$? Same question for $\alpha = 1/2$ and $\alpha > 1/2$.