Probability, homework 9, due December 10.

Exercise 1. Let $(X_k)_{k\geq 0}$ be i.i.d. random variables, $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ and $Y_m = \prod_{k=1}^m X_k$. Under which conditions is Y a $\mathcal{F}$-submartingale, supermartingale, martingale?

Exercise 2. Let $(S_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-martingale and τ a stopping time with finite expectation. Assume that there is a $c > 0$ such that, for all n, $\mathbb{E}(|S_{n+1} - S_n| \mid \mathcal{F}_n) < c$.
(i) Prove that $(S_{\tau\wedge n})_{n\geq 0}$ is a uniformly bounded martingale, and that $\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$.
(ii) Consider now the random walk $S_n = \sum_k^n X_k$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. For some $a \in \mathbb{N}^*$, let $\tau = \inf\{n \mid S_n = -a\}$. Prove that

$$\mathbb{E}(\tau) = \infty.$$

Exercise 3. As previously, consider the random walk $S_n = \sum_k^n X_k$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, $\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)$.
(i) Prove that $(S_n^2 - n, n \geq 0)$ is a $(\mathcal{F}_n)-$martingale. Let τ be a bounded stopping time. Prove that $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$.
(ii)Take now $\tau = \inf\{n \mid S_n \in \{-a, b\}\}$, where $a, b \in \mathbb{N}^*$. Prove that $\mathbb{E}(S_\tau) = 0$ and $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$. What is $\mathbb{P}(S_\tau = -a)$? What is $\mathbb{E}(\tau)$? Get the last result of the previous exercise by justifying the limit $b \to \infty$.

Exercise 4. Let $X_n, n \geq 0$, be iid complex random variables such that $\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty$. For some parameter $\alpha > 0$, let

$$S_n = \sum_{k=1}^n \frac{X_k}{k^\alpha}.$$

Prove that if $\alpha > 1/2$, S_n converges almost surely. What if $0 < \alpha \leq 1/2$?

Exercise 5. In a game between a gambler and a croupier, suppose that the total capital in play is 1. After the nth hand the proportion of the capital held by the gambler is denoted $X_n \in [0, 1]$, thus that held by the croupier is $1 - X_n$. We assume $X_0 = p \in (0, 1)$. The rules of the game are such that after n hands, the probability for the gambler to win the $(n + 1)$th hand is X_n; if he does, he gains half of the capital the croupier held after the nth hand, while if he loses he gives half of his capital. Let $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.
(i) Show that $(X_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$ martingale.
(ii) Show that $(X_n)_{n\geq 1}$ converges a.s. and in L^2 towards a limit Z.
(iii) Show that $\mathbb{E}(X_{n+1}^2) = \mathbb{E}(3X_n^2 + X_n)/4$. Deduce that $\mathbb{E}(Z^2) = \mathbb{E}(Z) = p$. What is the law of Z?
(iv) For any $n \geq 0$, let $Y_n = 2X_{n+1} - X_n$. Find the conditional law of X_{n+1} knowing $\mathcal{F}_n$. Prove that $\mathbb{P}(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n$, $\mathbb{P}(Y_n = 1 \mid \mathcal{F}_n) = X_n$ and express the law of Y_n.

(v) Let $G_n = \{Y_n = 1\}$, $P_n = \{Y_n = 0\}$. Prove that $Y_n \to Z$ a.s. and deduce that $\mathbb{P}(\liminf_{n\to\infty} G_n) = p$, $\mathbb{P}(\liminf_{n\to\infty} P_n) = 1 - p$. Are the variables $\{Y_n, n \geq 1\}$ independent ?
(vi) Interpret the questions c), d), e) in terms of gain, loss, for the gambler.

Exercise 6. Let $a > 0$ be fixed, $(X_i)_{i\geq 1}$ be iid, $\mathbb{R}^d$-valued random variables, uniformly distributed on the ball $\mathrm{B}(0, a)$. Set $S_n = x + \sum_{i=1}^n X_i$.
(i) Let f be a superharmonic function. Show that $(f(S_n))_{n\geq 1}$ defines a supermartingale.
(ii) Prove that if $d \leq 2$ any nonnegative superharmonic function is constant. Does this result remain true when $d \geq 2$?

Exercise 7 (bonus). Let X be a standard random walk in dimension 1, and for any positive integer a, $\tau_a = \inf\{n \geq 0 \mid X_{\tau_a} = a\}$. For any $\theta > 0$, calculate
$$\mathbb{E}\left((\cosh\theta)^{-\tau_a}\right).$$