Probability, homework 8, due November 26.

Exercise 1. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$. Show that $|\mathbb{E}(Y \mid \mathcal{G})| \leq \mathbb{E}(|Y| \mid \mathcal{G})$.

Exercise 2. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-algebra of $\mathcal{G}$. Show that $\mathbb{E}(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(Y \mid \mathcal{H})$.

Exercise 3. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, $(X_n)_{n\geq 0}$ a sequence of integrable random variables with $\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = 0$, and assume X_n is $\mathcal{F}_n$-measurable for every n. Let $S_n = \sum_{k=0}^{n} X_k$. Show that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

Exercise 4. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\mathcal{F}_T$ is a σ-algebra.

Exercise 5. Let S and T be stopping times for a filtration $(\mathcal{F}_n)_{n\geq 1}$. Prove that $\max(S, T)$ and $\min(S, T)$ are stopping times.

Exercise 6. Let $(X_n)_{n\geq 1}$ be a sequence of independent integrable random variables such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.
(i) Find sequences $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.
(ii) Assume moreover that there is a real number λ such that $e^{\lambda X_i} \in \mathrm{L}^1$ for any $i \geq 1$. Find a sequence $(a_n^{(\lambda)})_{n\geq 1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

Exercise 7 (bonus). You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1 - p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.