Probability, homework 7, due November 19.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be such that Ω is cointable and $\mathcal{A} = 2^\Omega$. Prove that almost sure convergene and convergence in probability are the same on this probability space.

Exercise 2. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1 + \dots + X_n}{X_1^2 + \dots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 3. Calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Hint: consider the sum of i.i.d. Poisson random variables. A Poisson random variable, with parameter λ, has values in $\mathbb{N}$ and $\mathbb{P}(X = \ell) = e^{-\lambda} \frac{\lambda^\ell}{\ell!}$.

Exercise 4. Let f be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \dots + x_n}{n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Exercise 5. Let $(X_i)_{i\geq 1}$ be i.i.d standard Cauchy random variables.
(i) Does $\frac{1}{n}\sum_{i=1}^n X_i$ converge almost surely?
(ii) Does $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converge in distribution?
(iii) Does $\frac{1}{n}\sum_{i=1}^n X_i$ converge in distribution?
(iv) Comment on these results in connection with the central limit theorem and the law of large numbers.

Exercise 6. The problem of the collector. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on $\{1, \dots, n\}$. Let $\tau_n = \inf\{m \geq 1 : \{X_1, \dots, X_m\} = \{1, \dots, n\}\}$ be the first time for which all values have been observed.
(i) Let $\tau_n^{(k)} = \inf\{m \geq 1 : |\{X_1, \dots, X_m\}| = k\}$. Prove that the random variables $(\tau_n^{(k)} - \tau_n^{(k-1)})_{2\leq k\leq n}$ are independent and calculate their respective distributions.
(ii) Deduce that $\frac{\tau_n}{n \log n} \to 1$ in probability as $n \to \infty$.

Exercise 7 (bonus). The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

(i) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)} = \sum_{\ell=1}^{n} X_i$ where the X_i's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

(ii) Prove that $\|B_n - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.