Probability, homework 6, due October 29.

Exercise 1. Let $(X_n)_{n\geq 1}$ be a sequence of Gaussian random variables, X_n having mean μ_n and variance σ_n^2. Assume $\mu_n \to \mu \in \mathbb{R}$ and $\sigma_n^2 \to \sigma^2 \in \mathbb{R}$. Prove that X_n converges in distribution to a Gaussian random variable with mean μ and variance σ^2.

Exercise 2. Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be real random variables, with X_n and Y_n independent for any $n \geq 1$, and assume that X_n converges in distribution to X and Y_n to Y. Prove that $X_n + Y_n$ converges in distribution to $X + Y$ (where X and Y are independent).

Exercise 3. Let f be a continuous fuction on $\mathbb{R}$, and assume that $(X_n)_{n\geq 1}$ converges to X in distribution. Prove that $(f(X_n))_{n\geq 0}$ converges to $f(X)$ in distribution.

Exercise 4. Find an example of real random variables $(X_n)_{n\geq 1}$, X, in L^1, such that $(X_n)_{n\geq 1}$ converges to X in distribution and $\mathbb{E}(X_n)$ converges, but not towards $\mathbb{E}(X)$.

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed real random variables, with $\mathbb{E}(X_1) = 0$, $\mathrm{var}(X_1) = 1$. Let $S_n = X_1 + \cdots + X_n$.

(i) Read the Kolmogorov 0-1 law (Theorem 10.6 in the book).

(ii) Prove that for any $A > 0$, $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\right) > 0$.

(iii) Prove that $\{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\} \in \cap_{n\geq 1}\sigma(X_i, i \geq n)$.

(iv) Deduce that $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty\right) = 1$.

Exercise 6

(i) Let X, Y be two independent and identically distributed real random variables. What is $\mathbb{P}(X = Y)$?

(ii) Let $(X_n)_{n\geq 1}$ be a sequence of real, independent and identically distributed random variables, with distribution function F. Show that almost surely we have

$$\max(X_1, \dots, X_n) \to \sup\{x \in \mathbb{R} \mid F(x) < 1\}.$$

Exercise 7 (bonus) Let $(X_n)_{n\geq 1}$ be a sequence of independent real random variables, all uniformly distributed on $[0, 1]$. Does $n \inf(X_1, \dots, X_n)$ converge in law as $n \to \infty$? If yes, what is the limiting distribution?