Probability, homework 3, due October 1.

Exercise 1. Calculate $\mathbb{E}(X)$ for the following probability measures $\mathbb{P}^X$.

(i) $\mathbb{P}^X$ has Gaussian density $\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$;
(ii) $\mathbb{P}^X$ has exponential dentity $\lambda e^{-\lambda x}\mathbb{1}_{x\geq 0}$ for some $\lambda > 0$;
(iii) $\mathbb{P}^X = p\delta_a + q\delta_b$ where $p + q = 1$, $p, q \geq 0$ and $a, b \in \mathbb{R}$;
(iv) $\mathbb{P}^X$ is the Poisson distribution: $\mathbb{P}^X(\{n\}) = e^{-\lambda}\frac{\lambda^n}{n!}$ for any integer $n \geq 0$, for some $\lambda > 0$.

Exercise 2. Let $(A_n)_{n\geq 0}$ be a set of pairwise disjoint events and $\mathbb{P}$ a probability. Show that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

Exercise 3. Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbb{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbb{1}_{[2,\infty)}(x)$$

What is the probability of the following events, $(-1/2, 1/2)$, $(-1/2, 3/2)$, $(2/3, 5/2)$, $(3, \infty)$?

Exercise 4. Let X be random variable on a countable probability space. Suppose that $\mathbb{E}(|X|) = 0$. Prove that $\mathbb{P}(X = 0) = 1$. Is it true, in general, that for any $\omega \in \Omega$ we have $X(\omega) = 0$?

Exercise 5. Let $\mathbb{P}$ be a probability measure on Ω, endowed with a σ-algebra $\mathscr{A}$.

(i) What is the meaning of the following events, where all A_n's are elements of $\mathscr{A}$?

$$\liminf_{n\to\infty} A_n = \bigcup_{n\geq 1}\bigcap_{k\geq n} A_k, \quad \limsup_{n\to\infty} A_n = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k.$$

(ii) In the special case $\Omega = \mathbb{R}$ and $\mathcal{A}$ is its Borel σ-algebra, for any $p \geq 1$, let

$$A_{2p} = \left[-1, 2 + \frac{1}{2p}\right), \quad A_{2p+1} = \left(-2 - \frac{1}{2p+1}, 1\right].$$

What are $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$?

(iii) Prove that the following always holds:

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n), \mathbb{P}\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{n\to\infty} \mathbb{P}(A_n).$$

Exercise 6. A simple given property is genetically encoded as a pair of alleles a and A, which yields three possible genotypes $\{A, A\})$, $\{a, a\}$, and $\{A, a\}$, represented in the population with respective probabilities p, q, r, $p + q + r = 1$, homogenously and with these same probabilities for each gender. A parent passes on one of its alleles, chosen at random uniformly, to its child; the genotype of the child combines alleles from both parents.

We assume that these coefficients p, q, r are stable from one generation to another. Explain why they actually depend on only one parameter:

$$\begin{cases} p = P^2, \\ q = Q^2, \\ r = 2PQ, \end{cases}$$

where $P + Q = 1$.

Exercise 7 (bonus). A samouraï has a strange idea. With his saber, he cuts a rigid spaghetti on two places, chosen uniformly and independently along this traditional pasta. What is the probability that he can create a triangle with sides these three pieces of spaghetti?