Probability, midterm.

Nine exercises perfectly solved yield the maximal possible grade. You therefore should read all of them first.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and A, B are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Exercise 2. Define an algebra and a σ-algebra. Let Ω be an infinite set (countable or not). Let $\mathcal{A}$ be the set of subsets of Ω that are either finite or with finite complement in Ω. Prove that $\mathcal{A}$ is an algebra but not a σ-algebra.

Exercise 3. Let X be a Poisson random variable with parameter λ. Prove that for any $r \in \mathbb{N}^*$, $\mathbb{E}(X(X-1)\dots(X-r+1)) = \lambda^r$. What is the variance of X?

Exercise 4. Let X be a positive random variable with density f. What is the density of $1/(1+X)$?

Exercise 5 Let X be a standard Gaussian random variable. Prove that for any $n \in \mathbb{N}^*$, $\mathbb{E}(X^{2n+1}) = 0$ and $\mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}$. You could for example use an expansion of the characteristic function of X.

Exercise 6. Let $S_n = \sum_{k=1}^n X_1$ where the X_i's are i.i.d. and $\mathbb{P}(X_1 = 1) = p$, $\mathbb{P}(X_1 = 0) = 1 - p$. Prove that for any $\varepsilon > 0$, $\mathbb{P}(S_n/n > p + \varepsilon) \leq e^{-\frac{1}{4}n\varepsilon^2}$.

Exercise 7. Let X and Y be two independent exponential random variables with parameter 1. What is the distribution of $X + Y$?

Exercise 8 Let X and Y be independent random variables uniform on $[0, 1]$. What is $\mathbb{E}(|X - Y|)$?

Exercise 9. Let the X_ℓ's be independent standard Cauchy random variables. Do $n^{-1}\sum_{\ell=1}^n X_\ell$ satisfy a law of large numbers? Do $n^{-1/2}\sum_{\ell=1}^n X_\ell$ satisfy a central limit theorem?

Exercise 10. Prove that if a sequence of real random variables (X_n) converge in distribution to X, and (Y_n) converges in distribution to a constant c, then $X_n + Y_n$ converges in distribution to $X + c$.

Exercise 11 Let the X_ℓ's be i.i.d. with mean 0 and variance $0 < \sigma^2 < \infty$. Does $n^{-\alpha}\sum_{\ell=1}^n X_\ell$ converge in distribution for $0 < \alpha < 1/2$? Same question for $\alpha = 1/2$ and $\alpha > 1/2$.

Exercise 12. Let X_ℓ and Y_ℓ be independent sequences of random variables, such that X_ℓ (resp. Y_ℓ) converge in distribution to X (resp. Y), with X and Y independent. Prove that $X_\ell + Y_\ell$ converges in distribution to $X + Y$.

Exercise 13. Prove that convergence in probability implies almost sure convergence along a subsequence.

Exercise 14 Prove that convergence in L^p implies convergence in probability.

Exercise 15 Let the X_ℓ's be i.i.d. with a Gaussian distribution, with mean 2 and variance 2. What is the limit of $(X_1^2 + \cdots + X_n^2)/(X_1 + \cdots + X_n)$ as $n \to \infty$? In which sense?

Exercise 16 Let the X_ℓ's be independent random variables, and $0 < c_1 < c_2$ be absolute constants. Let $\mu_\ell = \mathbb{E}(X_\ell)$, and $\sigma_\ell^2 = \mathrm{Var}(X_\ell)$ satisfy $c_1 < \sigma_\ell^2 < c_2$ for any ℓ. State and prove a central limit theorem for $\sum_{\ell=1}^n X_\ell$.