Probability, homework 6.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be such that Ω is countable and $\mathcal{A} = 2^\Omega$. Prove that almost sure convergence and convergence in probability are the same on this probability space.

Exercise 2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables, with respective distributions being Gaussian, with respective mean $\mu_n \in \mathbb{R}$ and variance $\sigma_n^2 > 0$. Prove that if X_n converges in distribution, then μ_n and σ_n^2 need to converge, and identify the limiting random variable.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. real random variables, with $\mathbb{E}(X_1) = 0$, $\mathrm{var}(X_1) = 1$. Let $S_n = X_1 + \cdots + X_n$.

a) Prove that for any $A > 0$, $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\right) > 0$.

b) Prove that $\{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\} \in \cap_{n\geq 1}\sigma(X_i, i \geq n)$.

c) Deduce that $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty\right) = 1$.

d) Prove that for any subsequence $(n_k)_{k\geq 1}$, $\mathbb{P}\left(\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k}} = +\infty\right) = 1$.

e) Prove that $(\frac{S_n}{\sqrt{n}})_{n\geq 1}$ does not converge in probability.

Exercise 4 *Central value of the partial exponential function.* Calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Hint: this is a probability related to a sum of independent Poisson random variables with parameter 1.

Exercise 5 *The number of buses stopping till time t.* Let $(X_n)_{n\geq 1}$ be i.i.d random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, X_1 being an exponential random variable with parameter 1. Define $T_0 = 0$, $T_n = X_1 + \cdots + X_n$, and for any $t > 0$,

$$N_t = \max\{n \geq 0 \mid T_n \leq t\}$$

a) For any $n \geq 1$, calculate the joint distribution of $(T_1, \dots, T_n)$.

b) Deduce the distribution of N_t, for arbitrary t.

Exercise 6 *Large deviations.* Let $(X_n)_{n\geq 1}$ be i.i.d random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, X_1 with mean μ, and

$$L(\lambda) = \begin{cases} \log \mathbb{E}(e^{\lambda X_1}) & \text{if } \mathbb{E}(e^{\lambda X_1}) < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

and $L^*(x) = \sup(x\lambda - L(\lambda) \mid \lambda \in \mathbb{R})$.

a) Check that for any $\lambda \in \mathbb{R}$, $L(\lambda) \geq \lambda\mu$.

b) Prove that for any $\alpha > 0$ and $n \geq 1$,

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} - \mu \geq \alpha\right) \leq e^{-nL^*(\mu+\alpha)}.$$

c) Prove that for any $\alpha > 0$ and $n \geq 1$,

$$\mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \alpha\right) \leq e^{-nL^*(\mu+\alpha)} + e^{-nL^*(\mu-\alpha)}.$$

d) Deduce the most general law of large numbers you can from the previous inequality. As a first step, you could for example calculate, L, L^* for ± 1 or Cauchy random variables and see what happens in the previous inequality.