Probability, homework 5.

Exercise 1. Let X be a standard Gaussian random variable. What is the density of $1/X^2$?

Exercise 2. In the (O,x,y) plane, a random ray emerges from a light source at point $(-1,0)$, towards the (O,y) axis. The angle with the (O,x) axis is uniform on $(-\frac{\pi}{2},\frac{\pi}{2})$. What is the distribution of the impact point with the (O,y) axis?

Exercise 3. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n\to\infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1+\cdots+x_n}{n}\right)\mathrm{d}x_1\dots\mathrm{d}x_n.$$

Exercise 4. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

a) Let $S_n(x)=B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)}=\sum_{\ell=1}^{n}X_i$ where the X_i's are independent and $\mathbb{P}(X_i=1)=x$, $\mathbb{P}(X_i=0)=1-x$. Prove that $B_n(x)=\mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n-f\|_{L^\infty([0,1])}\to 0$ as $n\to\infty$.

Exercise 5. The problem of the collector. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables uniformly distributed on $\{1,\dots,n\}$. Let $\tau_n=\inf\{m\geq 1:\{X_1,\dots,X_m\}=\{1,\dots,n\}\}$ be the first time for which all values have been observed.

a) Let $\tau_n^{(k)}=\inf\{m\geq 1:|\{X_1,\dots,X_m\}|=k\}$. Prove that the random variables $(\tau_n^{(k)}-\tau_n^{(k-1)})_{2\leq k\leq n}$ are independent and calculate their respective distributions.

b) Deduce that $\frac{\tau_n}{n\log n}\to 1$ in probability as $n\to\infty$, i.e. for any $\varepsilon>0$,

$$\mathbb{P}\left(\left|\frac{\tau_n}{n\log n}-1\right|>\varepsilon\right)\to 0.$$

Exercise 6. For any $d\geq 1$, we admit that there is only one probability measure μ on $\mathcal{S}_d$, (the $(d-1)$-th dimensional sphere embedded in $\mathbb{R}^d$) that is uniform, in the following sense: for any isometry $A\in\mathrm{O}(d)$ (the orthogonal group in $\mathbb{R}^d$), and any continuous function $f:\mathcal{S}_d\to\mathbb{R}$,

$$\int_{\mathcal{S}_d} f(x)\mathrm{d}\mu(x)=\int_{\mathcal{S}_d} f(Ax)\mathrm{d}\mu(x).$$

Let $X = (X_1, \ldots, X_d)$ be a vector of independent centered and reduced Gaussian random variables.

a) Prove that the random variable $U = X/\|X\|_{L^2}$ is uniformly distributed on the sphere.

b) Prove that, as $d \to \infty$, the main part of the globe is concentrated close to the Equator, i.e. for any $\varepsilon > 0$,

$$\int_{x \in \mathcal{S}_d, |x_1| < \epsilon} \mathrm{d}\mu(x) \to 1.$$