Probability, homework 3.

Exercise 1. Let X be a nonnegative random variable with null expectation. Prove that it is 0 almost surely.

Exercise 2. Let X be a random variable in $\mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n\geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_N) \underset{n\to\infty}{\longrightarrow} 0$. Prove that $\mathbb{E}(X\mathbb{1}_{A_n}) \underset{n\to\infty}{\longrightarrow} 0$.

Exercise 3. Let X be a Gaussian random variable with expectation 0 and variance σ^2. What is $\mathbb{E}(X^3)$? What is $\mathbb{E}(X^4)$?

Exercise 4. Let (X, Y) be chosen uniformly on the triangle $\{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x \geq 0, y \geq 0\}$. What is the density function of (X, Y)? Find the distributions of $X + Y$, $X - Y$, XY.

Exercise 5. A samouraï wants to create a triangle with a (rigid) spaghetti. With his saber, he cuts this spaghetti on two places, chosen uniformly and independently along this traditional pasta. What is the probability that he can create a triangle with sides these three pieces of spaghetti?

Exercise 6. Assume that $X_1, X_2, \dots$ are independent random variables uniformly distributed on $[0, 1]$. Let $Y^{(n)} = n \inf\{X_i, 1 \leq i \leq n\}$. Prove that it converges weakly to an exponential random variable, i.e. for any continuous bounded function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{E}\left(f(Y^{(n)})\right) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^+} f(u)e^{-u}\mathrm{d}u.$$