Probability, homework 1.

Exercise 1. Let $(\mathcal{G}_\alpha)_{\alpha\in A}$ be an arbitrary family of σ-algebras defined on an abstract space Ω. Show that $\cap_{\alpha\in A}\mathcal{G}_\alpha$ is also a σ-algebra.

Exercise 2. Let $\mathcal{A}$ be a σ-algebra. Prove that if, for all $n \in \mathbb{N}$, $A_n \in \mathcal{A}$, then $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are in $\mathcal{A}$.

Exercise 3. Prove the Bonferroni inequalities: if $A_i \in \mathcal{A}$ is a sequence of events, then

(i) $\mathbb{P}\left(\cup_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)$,
(ii) $\mathbb{P}\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k)$.

Exercise 4. Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. For $a \in \mathbb{Z}^*$, let $T_a = \inf\{n \geq 0 : s_n = a\}$. Prove that $\mathbb{E}(T_a) = \infty$.

Exercise 5. Let $(s_n)_{n\geq 0}$ be a 1-dimensional, unbiased random walk. Prove that $\mathbb{P}\left(\limsup_{n\to\infty} \frac{s_n}{\sqrt{n}} = \infty\right) = 1$.

Exercise 6. Let n and m be random numbers chosen independently and uniformly on $[\![1, N]\!]$. What are $\Omega, \mathcal{A}$ and $\mathbb{P}$ (which all implicitly depend on N) ? Prove that $\mathbb{P}(n \wedge m = 1) \underset{N\to\infty}{\longrightarrow} \zeta(2)^{-1}$ where $\zeta(2) = \prod_{p\in\mathcal{P}}(1 - p^{-2})^{-1} = \sum_{n\geq 1} n^{-2} = \frac{\pi^2}{6}$ (you don't have to prove these equalities). Here $\mathcal{P}$ is the set of prime numbers and $n \wedge m = 1$ means that their greatest common divisor is 1.