Dynamical Systems, homework 2.

Exercise 1. Let $f_\lambda(x) = x^2 + \lambda$.
a) Find some λ for which a tangent bifurcation occurs.
b) Discuss the stability of the fixed points near the bifurcation value.
c) If λ is chosen so that there is no fixed point, describe the orbits.
d) Describe the fixed points and their stability for $\lambda = -1/2$ and $\lambda = -1$.
e) Describe the bifurcation occurring at $\lambda = -3/4$.

Exercise 2. What kind of bifurcation (tangent, period-doubling, or none) occurs in the following examples?
a) $f_\lambda(x) = x^2 + x + \lambda$ at $\lambda = -1$.
b) $f_\lambda(x) = x^3 + \lambda x$ at $\lambda = -1$.
c) $f_\lambda(x) = x^3 + \lambda$ at $\lambda = \frac{2}{3\sqrt{3}}$.
d) $f_\lambda(x) = \lambda(e^x - 1)$ at $\lambda = -1$.
e) $f_\lambda(x) = x + x^3 + \lambda^2$ at $\lambda = 0$.

Exercise 3. For $f_\lambda(x) = x^5 - \lambda x^3$, what is the behavior of the 2 cycles for the bifurcation at $\lambda = 2$?

Exercise 4. The Hénon map is

$$T(x, y) = (1 - x^2 + y, x)$$

a) Find a point with prime period 2.
b) Is it stable?
c) What is its Lyapunov exponent?

Exercise 5. Get documented about the general version of Sarkovskii's theorem. If f is continuous, can the associated dynamical system have period 176 but no period 96? Same question if f is not assumed continuous.

Exercise 6. Assume f is continuous and that for some $n \geq 3$, f has a cycle

$$a_1 \stackrel{f}{\to} a_2 \stackrel{f}{\to} \dots \stackrel{f}{\to} a_n \stackrel{f}{\to} a_1$$

with $a_1 < \dots < a_n$. Prove that f has all possible primitive periods.

Exercise 7. Assume we apply Newton's method to find the zeros of the sine function. For $k \in \mathbb{Z}$, is the basin of attraction of $k\pi$ bounded ?

Exercise 8. Let $f_\lambda(x) = \lambda \sin x$. Describe the first three bifurcations as λ increases from 0.

Exercise 9. Let $f(x) = \pi \sin x$ be defined from $[0, \pi]$ to itself. Prove that the associated dynamical system is chaotic.

Exercise 10. Let $f(x) = 4x(1-x)$ be defined from $[0,1]$ to itself.

a) Thanks to the conjugation with the tent map, prove that $f^{\circ n}$ has 2^n fixed points.

b) What is the Lyapunov exponent of a periodic point of period n ?

Exercise 11. Give an example of polynomial with real coefficients whose Schwarzian derivative is not always negative.

Give an example of some n and some function f with n critical points and at least $n+3$ attracting cycles.