Complex analysis, homework 9, solutions.

Exercise 1. [18 points] Let C be the arc defined by

$$z(t) = \begin{cases} 3e^{i\pi t} & \text{if } 0 \leq t \leq 1, \\ -3 + 6(t-1) & \text{if } 1 \leq t \leq 2, \end{cases}$$

Evaluate the integral $\int_C f(z)\,\mathrm{d}z$ for the following functions f (give your answer in $x + iy$ form).

(1) $f(z) = \dfrac{\cos z}{(z+i)^2(z-4)}$;

(2) $f(z) = \dfrac{\cos z}{(z-i)^2(z-4i)}$;

(3) $f(z) = \dfrac{1}{(z-i)^2(z+2i)(z-2i)}$.

Solution. Note that C is a simple closed contour positively oriented (this is the boundary of the upper half disk about 0 with radius 3).

(1) f is analytic on $\mathbb{C} \setminus \{-i, 4\}$. In particular, f is analytic on and within C, so by Cauchy-Goursat theorem,

$$\int_C f(z)\,\mathrm{d}z = 0.$$

(2) f is analytic on $\mathbb{C} \setminus \{i, 4i\}$ and i is interior to C. So we set $g(z) = \frac{\cos z}{z-4i}$ which is analytic on and within C and apply Cauchy integral formula to get

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{g(z)}{(z-i)^2}\,\mathrm{d}z = 2i\pi g'(i).$$

Now note that

$$g'(z) = \frac{-\sin z}{z-4i} - \frac{\cos z}{(z-4i)^2}$$

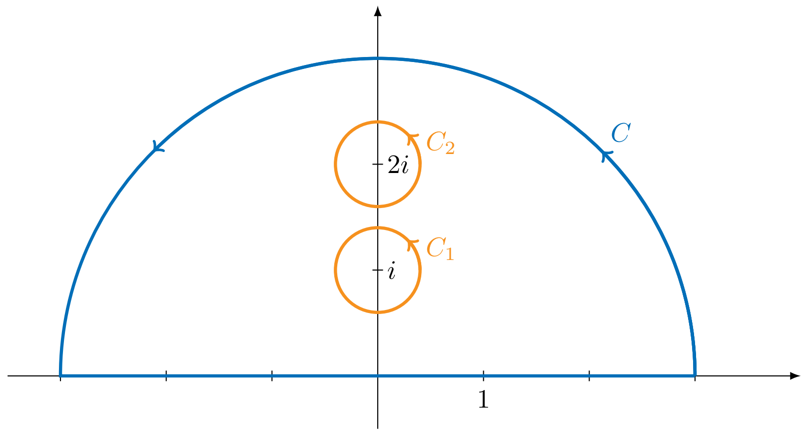
and therefore

$$g'(i) = \frac{-\sin i}{-3i} - \frac{\cos i}{(-3i)^2} = \frac{e^{i^2} - e^{-i^2}}{-6} - \frac{e^{i^2} + e^{-i^2}}{-18} = \frac{e - e^{-1}}{6} + \frac{e^{-1} + e}{18} = \frac{2e - e^{-1}}{9}.$$

So finally, we get

$$\int_C f(z)\,\mathrm{d}z = 2i\pi \frac{2e - e^{-1}}{9}.$$

(3) f is analytic on $\mathbb{C} \setminus \{i, 2i, -2i\}$ and i and $2i$ are interior to C. So we first introduce two contours C_1 and C_2 chosen as on the picture below.



Since f is analytic on and between C and C_1, C_2, we can apply the theorem of Section 53 to get

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z.$$

For the integral on C_1, we set $g(z) = \frac{1}{(z+2i)(z-2i)} = \frac{1}{z^2+4}$, which is analytic on and within C_1, and apply Cauchy integral formula to get

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_1} \frac{g(z)}{(z-i)^2}\,\mathrm{d}z = 2i\pi g'(i).$$

Now note that $g'(z) = -\frac{2z}{(z^2+4)^2}$ and therefore

$$g'(i) = -\frac{2i}{(i^2+4)^2} = -\frac{2i}{3^2} = -\frac{2i}{9}$$

So we get

$$\int_{C_1} f(z)\,\mathrm{d}z = \frac{4\pi}{9}.$$

For the integral on C_2, we set $g(z) = \frac{1}{(z-i)^2(z+2i)}$, which is analytic on and within C_2, and apply Cauchy integral formula to get

$$\int_{C_2} f(z)\,\mathrm{d}z = \int_{C_2} \frac{g(z)}{z-2i}\,\mathrm{d}z = 2i\pi g(2i) = 2i\pi\frac{1}{i^2\cdot 4i} = -\frac{\pi}{2}$$

So finally, we get

$$\int_C f(z)\,\mathrm{d}z = \frac{4\pi}{9} - \frac{\pi}{2} = -\frac{\pi}{18}.$$

Exercise 2. [6 points] Let $M, R > 0$. Let f be an analytic on and within the circle centered at 0 with radius R. Assume $|f(z)| \leq M$ for any $|z| \leq R$. Let n be a nonegative integer and $0 < \rho < R$. For $|z| \leq \rho$, find an upper bound for $|f^{(n)}(z)|$ which depends only on M, R, ρ, n.

Solution. Let $|z| \leq \rho$ and $r > 0$. Let C_r be the circle centered at z with radius r positively oriented. Note that C_r is included in the region $\{w : |w| \leq R\}$ if and only if $r \leq R - |z|$. In that case, f is analytic on and within C_r and $|f(w)| \leq M$ for any w on C_r: hence, by Cauchy's inequality

$$|f^{(n)}(z)| \leq \frac{Mn!}{r^n}.$$

Hence, we want to take r as large as possible to get the better bound possible. So we take $r = R - |z|$ and, in order to get a bound that does not depend on z, we use the lower bound $r \geq R - \rho$. Hence, we finally get

$$|f^{(n)}(z)| \leq \frac{Mn!}{(R-\rho)^n}.$$

Exercise 3. [6 points] Let f be an entire function. Assume there is a nonnegative integer n and a constant $M > 0$ such that $|f(z)| \leq M|z|^n$ for any $z \in \mathbb{C}$. Prove f is a polynomial.
Hint: You can first prove that $f^{(n+1)}(z) = 0$ using ideas similar to the proof of Liouville's theorem.

Solution. Let $z_0 \in \mathbb{C}$ and $R > 0$. Let C be the circle centered at z_0 with radius R positively oriented. Then, f is analytic on and within C. Moreover, for z on C, we have $|z| \leq |z - z_0| + |z_0| = R + |z_0|)$ and $|f(z)| \leq M|z|^n \leq M(R + |z_0|)^n$. Hence, by Cauchy's inequality

$$|f^{(n+1)}(z_0)| \leq \frac{M(R+|z_0|)^n(n+1)!}{R^{n+1}} = \frac{1}{R} \cdot M(n+1)! \left(1 + \frac{|z_0|}{R}\right)^n.$$

Now, we let $R \to \infty$ (letting z_0 fixed). In particular, if $R \geq |z_0|$, we have $1 + \frac{|z_0|}{R} \leq 2$ and so

$$|f^{(n+1)}(z_0)| \leq \frac{1}{R} \cdot M(n+1)!2^n.$$

This is arbitrarily small when R is sufficiently large, so we deduce that $|f^{(n+1)}(z_0)| = 0$. We proved that $f^{(n+1)}(z) = 0$ for any $z \in \mathbb{C}$. It follows that f is a polynomial.

Let's prove this last step. We proceed by induction on n to prove: for $n \geq 0$, if a function f satisfies $f^{(n+1)}(z) = 0$ for any $z \in \mathbb{C}$, then f is a polynomial of a degree at most n.

- **Basis step:** We take $n = 0$. Let f be a function such that $f'(z) = 0$ for any $z \in \mathbb{C}$. Then, since antiderivatives on a domain ($\mathbb{C}$ is a domain) are unique up to an additive constant and 0 is an antiderivative of 0, we get that f is a constant, that is a polynomial of degree at most 0.
- **Inductive step:** Let $k \geq 0$ be an integer. Assume the result is true for $n = k$, we want to prove it for $n = k + 1$. For this, let f be a function such that $f^{(k+2)}(z) = 0$ for any $z \in \mathbb{C}$. Let $g = f'$. Then $g^{(k+1)}(z) = 0$ for any $z \in \mathbb{C}$. So, by induction hypothesis, we get that g is a polynomial of degree at most k. We can write

$$g(z) = \sum_{j=0}^{k} a_j z^j,$$

for some $a_0, \ldots, a_k \in \mathbb{C}$. Hence, an antiderivative of g is

$$G(z) = \sum_{j=0}^{k} \frac{a_j}{j+1} z^{j+1} = \sum_{\ell=1}^{k+1} \frac{a_{\ell-1}}{\ell} z^{\ell}.$$

Since antiderivatives on a domain are unique up to an additive constant, we deduce that f equals G up to an additive constant, and therefore is a polynomial of degree at most $k+1$. This proves the result for $n = k+1$.

This concludes the proof by induction. We can therefore conclude the exercise by saying that f has to be a polynomial of a degree at most n.