Complex analysis, homework 7, solutions.

Exercise 1. [12 points] For each of the following arc C, sketch it and say if it is a simple arc, a simple closed curve, a smooth arc and/or a contour (that is for each one of the 4 previous properties, say if it holds or not). No justification required.

(1) Let C be the arc defined by

$$z(t) = \begin{cases} 2t - it & \text{if } 0 \le t \le 2, \\ 8 - 2i - 2t & \text{if } 2 \le t \le 3, \\ 8 - 8i + 2(i-1)t & \text{if } 3 \le t \le 4. \end{cases}$$

(2) Let C be the arc defined by

$$z(t) = t + it^2, -2 \le t \le 2.$$

(3) Let C be the arc defined by

$$z(t) = 1 + e^{2it}, 0 \le t \le 2\pi.$$

Solution.

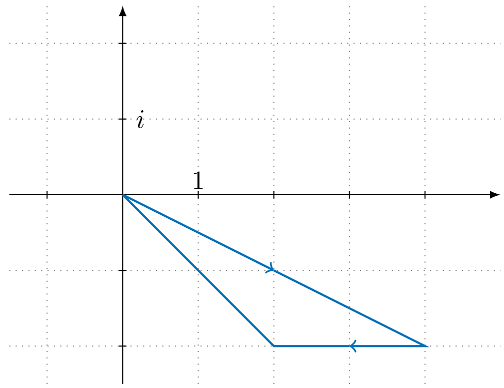
(1) This arc is
- not a simple arc: first and last point are the same.
- a simple closed curve: not twice the same point except the first and last point which are the same.
- not a smooth arc: $z(t)$ is not differentiable at 2 and 3 (this can be annoying to justify). A simple way to justify the arc is not smooth is to show $z'(t)$ is not continuous at 2 and at 3 (recall in order to be a differentiable arc, one needs both $z(t)$ is differentiable on $[a, b]$ **and** $z'(t)$ is continuous on $[a, b]$): indeed we have

$$z'(t) = \begin{cases} 2 - i & \text{if } 0 \le t \le 2, \\ -2 & \text{if } 2 \le t \le 3, \\ 2(i-1) & \text{if } 3 \le t \le 4. \end{cases}$$

- a contour: it is a piecewise smooth arc, composed of the following three arcs

$$z(t) = 2t - it, \quad 0 \le t \le 2,$$
$$z(t) = 8 - 2i - 2t, \quad 2 \le t \le 3,$$
$$z(t) = 8 - 8i + 2(i-1)t, \quad 3 \le t \le 4.$$

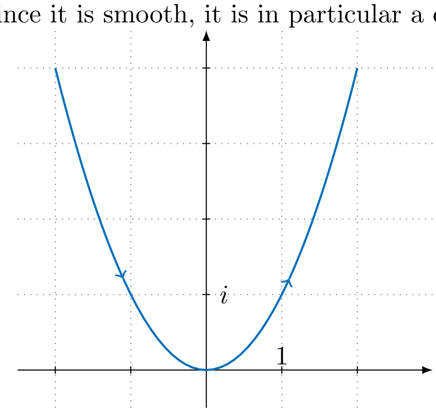
Each of these arcs is smooth: $z(t)$ is differentiable on the whole interval, $z'(t)$ is continuous on the whole interval and $z'(t) \neq 0$ on the interior of the interval (actually here on the whole interval).



(2) This arc is
- a simple arc: not twice the same point.
- not a simple closed curve: the first and last point which are different.
- a smooth arc: $z(t)$ is differentiable at any $t \in [-2, 2]$ and

$$z'(t) = 1 + 2it.$$

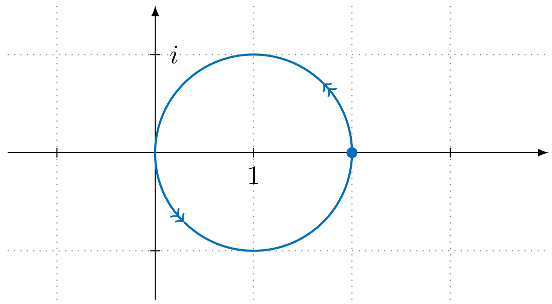
  Hence $z'(t)$ is continuous at any $t \in [-2, 2]$ and $z'(t) \neq 0$ at any $t \in (-2, 2)$.
- a contour: since it is smooth, it is in particular a contour.



(3) This arc is
- not a simple arc: this arc covers **twice** the circle centered at 1 with radius 1 (this can be sketched with double arrows, see below). Hence, it takes the same value twice so it is not simple.
- not a simple closed curve: first and last point are the same so it is a closed curve, but it takes other values twice so it is not a simple closed curve.
- a smooth arc: $z(t)$ is differentiable at any $t \in [0, 2\pi]$ and

$$z'(t) = 2ie^{2it}.$$

  Hence $z'(t)$ is continuous at any $t \in [0, 2\pi]$ and $z'(t) \neq 0$ at any $t \in (0, 2\pi)$.
- a contour: since it is smooth, it is in particular a contour.



Exercise 2. [6 points] Let C be the arc defined by

$$z(t) = \begin{cases} e^{-it} & \text{if } 0 \leq t \leq \pi, \\ t - 1 - \pi & \text{if } \pi \leq t \leq \pi + 2, \end{cases}$$

and $f(z) = 2\,\mathrm{Re}(z)$. Calculate the following integral

$$\int_C f(z)\,\mathrm{d}z.$$

Solution. We use the definition of contour integrals

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_0^{\pi+2} f(z(t))z'(t)\,\mathrm{d}t \\ &= \int_0^{\pi} f(e^{-it}) \cdot (-ie^{-it})\,\mathrm{d}t + \int_{\pi}^{\pi+2} f(t-1-\pi) \cdot 1\,\mathrm{d}t \\ &= \int_0^{\pi} 2\cos(-t) \cdot (-ie^{-it})\,\mathrm{d}t + \int_{\pi}^{\pi+2} 2(t-1-\pi)\,\mathrm{d}t, \end{aligned}$$

using $f(z) = 2\,\mathrm{Re}(z)$. For the first part, we use that $\cos(-t) = \cos t = \frac{e^{it}+e^{-it}}{2}$ to get

$$\begin{aligned} \int_0^{\pi} 2\cos(-t) \cdot (-ie^{-it})\,\mathrm{d}t &= -i\int_0^{\pi} (e^{it} + e^{-it})e^{-it}\,\mathrm{d}t = -i\int_0^{\pi} (1 + e^{-2it})\,\mathrm{d}t \\ &= -i\left[t + \frac{e^{-2it}}{-2i}\right]_0^{\pi} = -i\left(\pi + \frac{1}{-2i} - 0 - \frac{1}{-2i}\right) = -i\pi. \end{aligned}$$

For the other part, we set $s = t - 1 - \pi$ and get

$$\int_{\pi}^{\pi+2} 2(t-1-\pi)\,\mathrm{d}t = \int_{-1}^{1} 2s\,\mathrm{d}s = 0.$$

So finally

$$\int_C f(z)\,\mathrm{d}z = -i\pi.$$

Exercise 3. [6 points] Let C be the contour defined by $z(\theta) = e^{i\theta}$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Calculate the following integral

$$\int_C \mathrm{Log}(z)\,\mathrm{d}z.$$

Solution. We use the definition of contour integrals

$$\int_C f(z)\,\mathrm{d}z = \int_{\pi/2}^{3\pi/2} \operatorname{Log}(e^{i\theta}) \cdot ie^{i\theta}\,\mathrm{d}\theta$$

Now note that

$$\operatorname{Log}(e^{i\theta}) = \ln|e^{i\theta}| + i\operatorname{Arg}(e^{i\theta}) = \begin{cases} i\theta & \text{if } \frac{\pi}{2} \le \theta \le \pi, \\ i(\theta - 2\pi) & \text{if } \pi < \theta \le \frac{3\pi}{2}. \end{cases}$$

Therefore, we get

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_{\pi/2}^{\pi} i\theta \cdot ie^{i\theta}\,\mathrm{d}\theta + \int_{\pi}^{3\pi/2} i(\theta - 2\pi) \cdot ie^{i\theta}\,\mathrm{d}\theta \\ &= -\int_{\pi/2}^{3\pi/2} \theta e^{i\theta}\,\mathrm{d}\theta + 2\pi \int_{\pi}^{3\pi/2} e^{i\theta}\,\mathrm{d}\theta \end{aligned}$$

For the second term, we have

$$2\pi \int_{\pi}^{3\pi/2} e^{i\theta}\,\mathrm{d}\theta = \frac{2\pi}{i} \cdot \left[e^{i\theta}\right]_{\pi}^{3\pi/2} = -2i\pi(-i - (-1))) = -2\pi - 2i\pi.$$

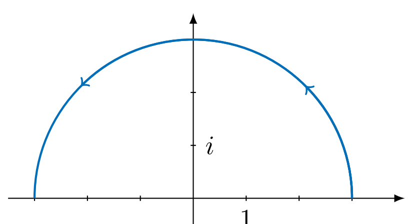
On the other hand, for the first term, integrating by part (we integrate $e^{i\theta}$ and differentiate θ),

$$\begin{aligned} \int_{\pi/2}^{3\pi/2} \theta e^{i\theta}\,\mathrm{d}\theta &= \left[\theta \frac{e^{i\theta}}{i}\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \frac{e^{i\theta}}{i}\,\mathrm{d}\theta = \left(\frac{3\pi}{2}\frac{e^{3i\pi/2}}{i} - \frac{\pi}{2}\frac{e^{i\pi/2}}{i}\right) + \int_{\pi/2}^{3\pi/2} ie^{i\theta}\,\mathrm{d}\theta \\ &= \left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) + \left[e^{i\theta}\right]_{\pi/2}^{3\pi/2} = -2\pi + (-i - i) = -2\pi - 2i. \end{aligned}$$

Therefore, we get

$$\int_C f(z)\,\mathrm{d}z = 2\pi + 2i - 2\pi - 2i\pi = i \cdot 2(1 - \pi).$$

Exercise 4. [6 points] Let C be the following arc (upper half circle centered at 0 with radius 3):



Prove the following bound

$$\left|\int_C \frac{z^2 - iz + 2}{z + 2}\,\mathrm{d}z\right| \le 42\pi.$$

Solution. The length of C is $\frac{1}{2} \cdot 2\pi 3 = 3\pi$. Then, using the triangle inequality, we get, for any z on C,

$$|z^2 - iz + 2| \le |z^2| + |-iz| + |2| = |z|^2 + |z| + 2 = 9 + 3 + 2 = 14,$$

using $|z| = 3$, and moreover

$$|z + 2| \ge |z| - 2 = 3 - 2 = 1.$$

Hence we get, for any z on C,

$$\left|\frac{z^2 - iz + 2}{z+2}\right| = \frac{|z^2 - iz + 2|}{|z+2|} \leq \frac{14}{1} = 14.$$

Finally, the theorem of Section 47 yields

$$\left|\int_C \frac{z^2 - iz + 2}{z+2}\,\mathrm{d}z\right| \leq 14 \cdot 3\pi = 42\pi.$$