Complex analysis, homework 3 due February 8th.

Exercise 1.[4 points] Calculate $(-2+2i)^{10}$. Give your result in the form $x+iy$ with x and y real numbers. Show you steps.
Remark: We have seen a method in class for this, do not expand directly $(-2+2i)^{10}$.

Exercise 2.[6 points]

(1) Find the fourth roots of i. Give them in exponential forms and then represent them on a picture. Highlight the principal fourth root.
(2) Find the third roots of $-8+8\sqrt{3}i$? Give them in exponential forms and then represent them on a picture. Highlight the principal third root.

Exercise 3.[4 points] We consider the following transformation $z \mapsto 2e^{i\pi/4}(z-1+i)$. Describe its effect on a point z of the complex plane in words (there should be three successive simple steps). Illustrate it with a picture in the case $z = 2+i$ (that is represent z and $2e^{i\pi/4}(z-1+i)$, as well as the results of the successive steps described earlier).

Exercise 4.[4 points] Prove that $\displaystyle\lim_{z\to 1-i} \frac{2z+1}{iz+1}$ exists and give its value in the form $x+iy$.

Exercise 5.[5 points] Let f be a function defined on $\mathbb{C}$. We say that f is Lipschitz on $\mathbb{C}$ if there exists $K>0$ such that, for any $z, z' \in \mathbb{C}$,

$$|f(z) - f(z')| \leq K|z-z'|.$$

Prove that, if f is Lipschitz on $\mathbb{C}$, then f has a limit at any point in $\mathbb{C}$.

Exercise 6.[5 points] Prove that $\displaystyle\lim_{z\to -1} \mathrm{Arg}(z)$ does not exist.

Exercise 7.[8 points] Let $z_0 \in \mathbb{C}$. **Prove or disprove** the following statements:

(1) Let f and g be functions defined on a deleted neighborhood of z_0.

$$\text{If } \lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty, \text{ then } \lim_{z\to z_0} (f(z)+g(z)) = \infty.$$

(2) Let f and g be functions defined on a deleted neighborhood of z_0.

$$\text{If } \lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty, \text{ then } \lim_{z\to z_0} (f(z)\times g(z)) = \infty.$$

Remark: In order to disprove a result, you have to give a counterexample.